which may be solved by quadrature,

$$I_5 = a \int_a^\infty u^{-2}F(u)\, du = \tfrac{1}{2}(\pi/2)^{1/2}\{\cos a^2 - \sin a^2 - 2a[C(a) + S(a)]\}.$$

Finally, the most significant integral from the point of view of the physical problem

$$I_6 = \int_0^\infty \{[C(x + a) - C(x - a)]^2 + [S(x + a) - S(x - a)]^2\}\, dx$$
$$= I_3 + I_4 - 2I_5 = \pi a,$$

a remarkably simple result.

NOTE ON THE POINCARÉ BOUNDARY-VALUE PROBLEM*

By E. E. JONES (*University of Nottingham, England*)

1. This note is concerned with the solution of a modified form of the Poincaré boundary-value problem [1]. It is required to solve the Poisson differential equation $\nabla^2\phi_i = f(x, y)$ for $\phi_i(x, y)$ defined in S_i, the region enclosed by the circle C of equation $|z| = a$, $(z = x + iy)$, such that on C

$$k \frac{\partial \phi_i}{\partial n} + l \frac{\partial \phi_i}{\partial s} + m\phi_i = g(x, y), \tag{1}$$

where k, l, m are constants, $f(x, y)$ is prescribed in S_i, $g(x, y)$ is prescribed on C, and $\partial/\partial n$, $\partial/\partial s$ denote differentiations along the inward normal and positive tangential directions respectively to C.

It is assumed that $\phi_i = \phi_{i0} + \Phi_i$, where $\phi_{i0}(x, y)$ is a particular integral of the Poisson equation reflecting all the singularities of the complete solution ϕ_i, and $\Phi_i(x, y)$ is harmonic in S_i, being together with its first partial derivatives single-valued and continuous in S_i. It is thus possible to write $\Phi_i = re\, W_i(z)$, where $W_i(z)$ is a regular function of z in S_i. If $z = ae^{i\theta} = \zeta$, $\bar{z} = a^2/\zeta$ on C, and by definition

$$h(\zeta) = -g - k \frac{\partial \phi_{i0}}{\partial r} + \frac{l}{a} \frac{\partial \phi_{i0}}{\partial \theta} + m\phi_{i0}, \tag{2}$$

then

$$h_i(z) = \frac{1}{2\pi i} \int_C \frac{(\zeta + z)h(\zeta)}{\zeta - z}\, d\zeta, \tag{3}$$

is the Schwarz integral representation of a function $h_i(z)$ regular in S_i with a real part equal to $h(\zeta)$ on C—for this to be so it is necessary for $h(\zeta)$ to satisfy the Lipschitz condition on C [2]. The boundary condition (1) then takes the form

$$re \left\{ \frac{\zeta}{\alpha} \frac{dW_i(\zeta)}{d\zeta} - mW_i(\zeta) - h_i(\zeta) \right\} = 0, \tag{4}$$

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where $\alpha = a/(k - il)$. When $re(m\alpha) < 0$, then a unique solution exists for $W_i(z)$ in S_i in the form

$$W_i(z) = \alpha \int_0^1 h_i(zt)t^{-m\alpha-1}\,dt. \tag{5}$$

This form of expression has an advantage over that given in [1], since it also includes the conditions for the existence of a solution. Thus when $m = 0$ the equation (5) determines $W_i(z)$ to within an arbitrary constant, provided the integral is uniformly convergent, which necessitates $h_i(0) = 0$. It thus follows that since the integral of (5) is uniformly convergent for $re(m\alpha) < 0$, and for $m = 0$ if $h_i(0) = 0$, then $W_i(z)$ is a regular function of z in S_i, as required, and $W_i'(z)$ can be obtained by differentiation under the integral sign [3].

Repeated integration by parts extends the range to all positive values of $re(m\alpha)$, including $m\alpha$ a positive integer—in this latter case solutions exist but are not uniquely determined [1].

2. The particular integral ϕ_{i0} can be determined by integrating the Poisson differential equation in the form

$$4\,\frac{\partial^2\phi_{i0}}{\partial z\,\partial\bar{z}} = f[x(z,\bar{z}),\, y(z,\bar{z})],$$

and the solution can be written as

$$\phi_{i0} = \phi_{i0}^* + re\,\Omega_i(z),$$

where $\Omega_i(z)$ is a function of z, regular outside C, having prescribed singularities in S_i only. It follows that (2) can be written as

$$h(\zeta) = h^*(\zeta) + re\left\{-\frac{a^2}{\bar{\alpha}\zeta}\,\bar{\Omega}_i'\!\left(\frac{a^2}{\zeta}\right) + m\bar{\Omega}_i\!\left(\frac{a^2}{\zeta}\right)\right\}.$$

But $h(\zeta) = \{re\,h_i(z)\}_C$, then evidently

$$h_i(z) = h_i^*(z) - \frac{a^2}{\bar{\alpha}z}\,\bar{\Omega}_i'\!\left(\frac{a^2}{z}\right) + m\bar{\Omega}_i\!\left(\frac{a^2}{z}\right),$$

which is a regular function of z in S_i, as required. On substitution into (5), and integrating by parts, then readily

$$W_i(z) = W_i^*(z) + \frac{\alpha}{\bar{\alpha}}\,\bar{\Omega}_i\!\left(\frac{a^2}{z}\right) + \frac{2mk\alpha^2}{a}\int_1^\infty \bar{\Omega}_i\!\left(\frac{a^2t}{z}\right)t^{m\alpha-1}\,dt. \tag{6}$$

This result determines $W_i(z)$ when the singularities of $\Omega_i(z)$ are prescribed, and extends the results of Ludford et al. [4].

A similar type of reasoning leads to the conclusion that it is possible to add the real part of any function of z regular in S_i to ϕ_{i0} without changing the value of ϕ_i.

3. For the external problem with ϕ_e defined in S_e, the region $|z| > a$, and satisfying a boundary condition similar to (1) with $\partial/\partial n$ denoting differentiation along the normal to C drawn into S_e, the solution analogous to (6) is

$$W_e(z) = W_e^*(z) + \frac{\bar{\alpha}}{\alpha}\,\bar{\Omega}_e\!\left(\frac{a^2}{z}\right) + \frac{2mk\bar{\alpha}^2}{a}\int_0^1 \Omega_e\!\left(\frac{a^2t}{z}\right)t^{-m\bar{\alpha}-1}\,dt. \tag{7}$$

Here

$$W_e^*(z) = -\bar{\alpha}\int_1^\infty h_e^*(zt)t^{m\bar{\alpha}-1}\,dt,$$

where $h_e^*(z)$ is regular in S_e, including infinity, such that on C

$$re\ h_e^*(z) = g(x, y) - k \frac{\partial \phi_{e0}^*}{\partial r} - \frac{l}{a} \frac{\partial \phi_{e0}^*}{\partial \theta} - m\phi_{e0}^*,$$

and $\Omega_e(z)$ is regular in S_i, having singularities only in S_e.

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ON MATRIX DIFFERENTIAL EQUATIONS*

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The purpose of this note is to obtain a necessary condition and a sufficient condition, of an algebraic nature, for the matrix differential equation

$$AX'' + XB = C, \tag{1}$$

where all matrices considered are n-square, to have a single-valued solution. Capital letters denote matrices and prime denotes the derivative with respect to x. The elements of A, B, C belong to the polynomial domain $\mathfrak{F}[x]$ of the field $\mathfrak{F}$ of real numbers. I is the identity matrix.

Theorem 1. If a solution matrix X of Eq. (1) exists, then the following pair of matrices

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}, \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{2}$$

are equivalent.

Proof. Clearly

$$\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -X'' \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & C \quad -AX'' \quad -XB \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{3}$$

and so the matrices of (2) are equivalent.

Theorem 2. If the matrices of (2) are similar and there exist non-singular constant matrices P, Q such that PAP^{-1}, QBQ^{-1} are diagonal matrices exhibiting the invariant factors a_i, $i = 1, 2, \cdots, \alpha$, and b_j, $j = 1, 2, \cdots, \beta$ of A, B, respectively, along the

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