

A NOTE ON UNIQUENESS IN CLASSICAL ELASTODYNAMICS*

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Introduction. The conventional uniqueness theorems of classical elasticity theory, first established by Kirchhoff [1]¹ for the equilibrium case and later extended to elastodynamics by Neumann [2], rest on the assumption that the elastic constants satisfy inequalities which are necessary and sufficient for the positive definiteness of the strain-energy density. The theorems referred to thus presuppose that

$$\mu > 0, \quad -1 < \sigma < \frac{1}{2}, \quad (1)$$

if μ and σ respectively denote the shear modulus and Poisson's ratio of the material.

Actual elastic materials obeying the linear isotropic stress-strain law conform to (1) and, in fact, fail to exhibit negative Poisson ratios. Nevertheless, the question as to whether conditions (1) are *necessary* for uniqueness, is of obvious theoretical interest. It is the purpose of the present note to show, with limitation to the first boundary-value problem of elastodynamics (surface displacements prescribed) and for bounded domains, that the inequalities (1) may be relaxed without loss of uniqueness. Specifically, we find that in this instance (1) may be replaced by the less stringent assumption

$$\mu \geq 0, \quad -\infty < \sigma < \frac{1}{2}, \quad 1 \leq \sigma < \infty, \quad (2)$$

or, equivalently, by

$$c_1^2 = \frac{\mu}{\rho} \geq 0, \quad c_2^2 = \frac{2(1-\sigma)\mu}{(1-2\sigma)\rho} \geq 0, \quad (3)$$

in which c_1 and c_2 are the velocities of equivoluminal and irrotational waves in a medium of unlimited extent, whereas ρ stands for the positive mass density.² Accordingly, the generalized inequalities (2) admit a simple intrinsic interpretation: they require that the two basic wave-velocities of the elastic solid be real.

The foregoing extension of Neumann's theorem, which will be proved subsequently, is the dynamic counterpart of a known extension of the uniqueness theorem in elastostatics. The solution to the first boundary-value problem in the equilibrium theory is unique (under suitable regularity assumptions) if, instead of (1), one stipulates merely that

$$\mu \neq 0, \quad -\infty < \sigma < \frac{1}{2}, \quad 1 < \sigma < \infty. \quad (4)$$

The relaxed inequalities (4) are evidently met provided the two wave velocities c_1 and c_2 are real and non-zero. For a bounded domain, this generalization of Kirchhoff's uniqueness theorem is implicit already in observations due to Kelvin [3]; it may also be inferred from available results appropriate to the Dirichlet problem for strongly

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¹Numbers in brackets refer to the list of publications at the end of this paper.

²Actually, the inequalities (3) are sufficient for uniqueness in the present circumstances even if ρ is negative.

elliptic systems of partial differential equations,³ since (4) hold if and only if the displacement equations of equilibrium are of the strongly elliptic type. As shown by Duffin and Noll [6],⁴ the inequalities (4) assure the uniqueness of the solution to the first problem of elastostatics also for unbounded exterior domains, provided the displacements are required to vanish uniformly at infinity. Finally, it is clear from counter-examples due to Ericksen and Toupin [8] and Ericksen [9] that (4), in the equilibrium case, cannot be weakened further without loss of uniqueness.

A generalization of the uniqueness theorem in elastodynamics. The displacement equations of motion in the linear theory of homogeneous and isotropic elastic solids take the form

$$\mu \nabla^2 \mathbf{u} + \frac{\mu}{1 - 2\sigma} \nabla \nabla \cdot \mathbf{u} + \mathbf{f} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2}, \quad (5)$$

in which $\mathbf{u}(x, t)$ is the displacement vector⁵ and $\mathbf{f}(x, t)$ is the body-force intensity vector, the single argument x representing the triplet of cartesian coordinates and t denoting the time. Equation (5), which must hold throughout the region of space D occupied by the medium, is subject to the initial conditions

$$\mathbf{u}(x, 0) = \mathbf{u}^0(x), \quad \left. \frac{\partial \mathbf{u}}{\partial t} \right|_{(x, 0)} = \mathbf{v}^0(x) \quad \text{in } D, \quad (6)$$

where $\mathbf{u}^0(x)$ and $\mathbf{v}^0(x)$ are the prescribed initial distributions of displacement and velocity. In the first problem of elastodynamics the accompanying boundary conditions are characterized by

$$\mathbf{u}(x, t) = \mathbf{u}^*(x, t) \quad \text{on } B \quad (0 < t < \infty), \quad (7)$$

if B is the boundary of D , while $\mathbf{u}^*(x, t)$ are the given surface displacements.

We now prove the following theorem. *Let D be a bounded regular region of space⁶ with the boundary B and suppose that (3) hold. Then there exists at most one $\mathbf{u}(x, t)$, twice continuously differentiable for (x_1, x_2, x_3) in $D + B$ ($0 \leq t < \infty$), which satisfies (5) in D ($0 < t < \infty$), and meets (6), (7).*

In view of the linearity of (5), (6), (7) it is evidently sufficient to show that

$$\mu \nabla^2 \mathbf{u} + \frac{\mu}{1 - 2\sigma} \nabla \nabla \cdot \mathbf{u} = \rho \frac{\partial^2 \mathbf{u}}{\partial t^2} \quad \text{in } D \quad (0 < t < \infty), \quad (8)$$

together with

$$\mathbf{u}(x, 0) = \left. \frac{\partial \mathbf{u}}{\partial t} \right|_{(x, 0)} = 0 \quad \text{in } D, \quad (9)$$

$$\mathbf{u}(x, t) = 0 \quad \text{on } B \quad (0 < t < \infty), \quad (10)$$

imply $\mathbf{u}(x, t) = 0$ in D ($0 < t < \infty$).

Let $K(t)$ be the kinetic energy of the entire body, so that

$$K(t) = \frac{\rho}{2} \int_D \left[\frac{\partial \mathbf{u}}{\partial t} \right]^2 dV, \quad (11)$$

³See Browder [4] and Morrey [5].

⁴See also [7].

⁵Letters in boldface designate vectors.

⁶The term "regular region of space" is used in the sense of Kellogg [10].

and define $G(t)$ through

$$G(t) = \frac{\mu}{2} \int_D [\kappa(\nabla \cdot \mathbf{u})^2 + (\nabla \times \mathbf{u})^2] dV, \quad \kappa = \frac{2(1 - \sigma)}{1 - 2\sigma}. \quad (12)$$

From the divergence theorem, with the aid of (8), (11), (12), and elementary vector-analytic identities, follows

$$\mu \int_B \left[\kappa \frac{\partial \mathbf{u}}{\partial t} \nabla \cdot \mathbf{u} + \frac{\partial \mathbf{u}}{\partial t} \times (\nabla \times \mathbf{u}) \right] \cdot \mathbf{n} dA = \frac{dK}{dt} + \frac{dG}{dt}, \quad (13)$$

where \mathbf{n} is the outward unit normal of B .

According to (13) and (10),

$$K(t) + G(t) = c \quad (0 < t < \infty), \quad (14)$$

in which c is a constant. But by (11), (12), (9),

$$K(0) = G(0) = 0, \quad (15)$$

whence, from (14) and continuity,

$$K(t) + G(t) = 0 \quad (0 \leq t < \infty). \quad (16)$$

Since $\kappa \geq 0$ by virtue of (3), both $K(t)$ and $G(t)$ are either non-negative or non-positive, so that (16) yields

$$K(t) = 0, \quad G(t) = 0 \quad (0 \leq t < \infty). \quad (17)$$

On the other hand, the first of (17) in conjunction with (11) leads to

$$\frac{\partial \mathbf{u}}{\partial t} = 0 \quad \text{in } D \quad (0 \leq t < \infty), \quad (18)$$

from which, because of (9), $\mathbf{u}(x, t) = \mathbf{u}(x, 0) = 0$ in $D(0 \leq t < \infty)$. This completes the proof.

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