ON THE TIME DERIVATIVE OF TENSORS IN MECHANICS OF CONTINUA*

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1. Introduction. In describing the motion of a continuum, not only have spatial coordinates been employed, but sometimes the continuum at time t is referred to a convected or a fixed reference frame. Since tensor fields, such as stress and strain, which arise in the mechanics of continua generally have different transformation laws, the time derivative of these tensors depends not only upon the particular reference frame chosen, but is intimately tied in with the constitutive equations of the medium. Moreover, while the covariant derivative and the time derivative of a tensor commute in a *material* coordinate system, such an interchangeability of order of derivatives is not permissible in *spatial* or even in a *convected* coordinate system.

The question of the time derivative of tensors, especially with reference to the stress tensor, i.e. the *stress rate*, has aroused much interest recently and has been discussed and investigated by Truesdell [1, 2], Noll [3], Oldroyd [4], Thomas [5], Cotter and Rivlin [6], and Prager [7]; the last, with the limitation to rectangular Cartesian coordinates, contains an account of the various definitions for the stress rate proposed in [1, 3, 4, 5, 6], as well as in others, including the earlier work of Jaumann [8][1]. Although the expressions for the stress rate deduced in [1–6] differ from one another, their underlying objective is the requirement that the constitutive equations of the medium involving the stress rate must remain invariant under rigid motions. In addition, Prager [7] has pointed out the desirability for a definition of the stress rate, the vanishing of which will render the invariants of the stress tensor stationary. Also, mention should be made of a very recent paper by Sedov [9], dealing with the time derivative of tensors, which will be referred to again.

The main purpose of the present investigation is to set forth a single general expression for the time derivative (or the time rate) of tensors, valid in all curvilinear coordinates, and applicable to any tensor fields (representing the variables of state in mechanics of continua) whose transformation law is known *a priori*. Indeed, the time derivative of a tensor as derived here is necessarily dependent on the transformation connections (which occur in the transformation law) of the tensor in question, and in particular obeys the same transformation law as the tensor itself. Furthermore, if the rate of a tensor (such as the stress rate), is defined in the context of the present paper, then (for a suitable representation of stress) the following conditions are fulfilled: (a) the stress rate vanishes when the medium executes rigid motion alone and when the stress is referred to a coordinate system participating in this motion, and (b) the rate of invariants of the stress tensor is stationary when the stress rate vanishes. This latter

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[1]As pointed out in [7], Noll's result [3] is identical to that given much earlier by Jaumann [8].

requirement is equivalent to that stipulated by Prager [7] when the coordinate system is rectangular Cartesian and the time derivative is taken in the sense of Jaumann.

Following some preliminary background in Sec. 2, which includes certain results needed in the subsequent analysis, the expression for the time derivative of tensors mentioned above is deduced in Sec. 3 [Eq. (3.13)], where its various parts are conveniently arranged in terms of symbols reminiscent of Christoffel symbols, and is applied to strain as well as stress. The reduction of the latter (i.e., the stress rate) to special cases and its comparison with the previous works, discussed in Sec. 4, are confined, for economy of space, to those in [2, 3, 4, 8, 9]. Finally, in analogy with the time derivative (3.13), an expression for a *coordinate derivative* of tensors is deduced in Sec. 5, which has the properties that (i) when applied to a tensor, it has the same transformation law as the tensor itself, and (ii) it commutes with the time derivative operator (3.13) in all coordinate systems. Aside from its possible future utility, this coordinate derivative of a tensor [Eq. (5.17)] is included here because of the manner in which its development parallels that of (3.13).

2. Preliminaries. Let a generic point of a continuum τ occupy initially (at time $t = t_0$) a position P with the material coordinates X^A, and at time t a position P' with coordinates X'^A; both positions, which may also be identified by position vectors $\mathbf{R}(X)$ and $\mathbf{r}(X', t)$, respectively, are measured relative to the same arbitrary but fixed curvilinear reference frame X. It is clear that while X^A and X'^A refer to the same generic point of the continuum at two different times, they represent two distinct positions in space separated by a displacement $\mathbf{u}$ such that

$$\mathbf{r} = \mathbf{R} + \mathbf{u}. \tag{2.1}$$

If we also introduce an arbitrary curvilinear reference frame x and denote the coordinates of P' in this frame by x^i, then the spatial coordinates x^i and X'^A refer to the same generic point of τ at time t, and the motion of the continuum relative to the fixed frame of reference X may be represented by

$$x^i = x^i(X^B, t); \qquad \left| \frac{\partial x^i}{\partial X^B} \right| > 0, \tag{2.2}$$

or by transformations of the type

$$\begin{aligned} x^i &= x^i(X'^A, t), \\ X'^A &= X'^A(X^B, t), \end{aligned} \tag{2.3}$$

with the restriction that their Jacobians be positive. When only the mapping (2.2) and its inverse are admitted, then the kinematics of continua may be presented as in [10], a description which is also adopted here[2]. Except for some modifications, the notation employed is similar to that in [10]; here, however, covariant differentiation is designated by a stroke (|), while comma is reserved for partial differentiation.

Because of (2.3), tensor transformations from x^i to X'^A have the transformation connections

$$x^i_{,A} \equiv \frac{\partial x^i}{\partial X'^A}, \qquad X'^A_{,i} \equiv \frac{\partial X'^A}{\partial x^i},$$

[2]As will become apparent later, the coordinates X'^A are introduced mainly for added clarity, although it will also offer some advantages.

whereas the tensor transformation laws between x^i and X^A may involve $X^A_{,i}$, r^A_i [see Eqs. (2.7)] or both, as well as their inverses. In what follows, it is convenient to select any one of the three sets (X^A, t), (X'^A, t), and (x^i, t) as independent variables. Thus, if ψ is representative of a tensor of any order, the partial derivatives of ψ with respect to the three space variables are related by expressions of the type

$$\psi_{,i} = \psi_{,A}X^A_{,i} = \psi_{,A'}X'^A_{,i} \ . \tag{2.4}$$

On the other hand, since the time t is common to all three sets of variables, in taking partial derivative with respect to time, it may be difficult to recognize which of the three sets of variables is taken as independent. To avoid this ambiguity when writing partial derivative with respect to time, it is advisable to specify which coordinates are being held fixed by attaching an appropriate subscript to the partial derivative in question. Accordingly, the partial derivatives of ψ with respect to t will be displayed as

$$\left(\frac{\partial\psi}{\partial t}\right)_X = \psi_{,A'}\left(\frac{\partial X'^A}{\partial t}\right)_X + \left(\frac{\partial\psi}{\partial t}\right)_{X'} \tag{2.5a}$$

$$= \psi_{,i}\left(\frac{\partial x^i}{\partial t}\right)_X + \left(\frac{\partial\psi}{\partial t}\right)_x , \tag{2.5b}$$

and similar expressions for $(\partial\psi/\partial t)_{X'}$ and $(\partial\psi/\partial t)_x$.

For future reference, we recall here that with the use of (2.1) and (2.4) the base vectors of the coordinate curves are related by

$$\begin{aligned} \mathbf{g}_i &= r^A_i\mathbf{G}_A = X'^A_{,i}\mathbf{G}'_A \ , \\ \mathbf{g}^i &= r^i_A\mathbf{G}^A = x^i_{,A'}\mathbf{G}'^A, \end{aligned} \tag{2.6}$$

where

$$r^i_A = x^i_{,A} - u^i \mid_A , \qquad r^A_i = X^A_{,i} + U^A \mid_i . \tag{2.7}$$

Further, if h denotes an arbitrary vector with components h^i and H^A in the coordinate systems x^i and X^A, respectively, then by (2.6)

$$h^i = r^i_A H^A, \tag{2.8a}$$

while

$$h^i \mid_j = X^B_{,j} r^i_A H^A \mid_B . \tag{2.8b}$$

With the notation

$$\begin{aligned} T^A_B &= \delta^A_B + U^A \mid_B , \\ R^i_j &= \delta^i_j - u^i \mid_j , \end{aligned} \tag{2.9}$$

where δ^i_j is the Kronecker delta, it follows also from (2.7) that

$$\begin{aligned} r^A_i &= T^A_B X^B_{,i} , \\ r^i_A &= R^i_j x^j_{,A} . \end{aligned} \tag{2.10}$$

It is instructive to consider here the velocity of a generic point of the continuum τ. If we select the set (X^A, t) as independent and specify the position vector of this point at time t by $\mathbf{r} = \mathbf{r}(X^A, t)$, and if we follow the motion of this point (by holding X^A fixed), then the velocity vector is simply

$$\mathbf{v} = \left(\frac{\partial \mathbf{r}}{\partial t}\right)_X . \tag{2.11}$$

If, on the other hand, the set (X'^A, t) is taken as independent, in which case $\mathbf{r} = \mathbf{r}(X'^A, t)$, then for an assigned value of X'^A (which may be the final position in space), $\mathbf{r} = \text{const.}$, $(\partial\mathbf{r}/\partial t)_{X'} = 0$, and by (2.5),

$$\mathbf{v} = \left(\frac{\partial \mathbf{r}}{\partial t}\right)_X = V'^A \mathbf{G}'_A = \left(\frac{\partial X'^A}{\partial t}\right)_X \mathbf{G}'_A , \tag{2.12}$$

which is an expected result.

While (2.12) represents the *total* (or absolute) velocity of P' relative to the X frame, the total velocity of P' relative to the x frame of reference is not $(\partial x^i/\partial t)_X$, but is given by

$$v^i = \left(\frac{\partial x^i}{\partial t}\right)_X - \left(\frac{\partial x^i}{\partial t}\right)_{X'} \tag{2.13}$$

which is easily verified by application of (2.5a) to x^i. The first term on the right-hand side of (2.13) is the velocity of P' relative to the x frame and will be denoted by

$$q^i = \left(\frac{\partial x^i}{\partial t}\right)_X , \tag{2.14}$$

and in order to assign an interpretation to the second term on the right-hand side of (2.13), we first observe that

$$\left(\frac{\partial x^i}{\partial t}\right)_{X'} = -x^i_{,A'}\left(\frac{\partial X'^A}{\partial t}\right)_x , \tag{2.15}$$

where $(\partial X'^A/\partial t)_x$ is easily interpreted as the absolute velocity of a point fixed in the x frame (i.e., a point, the x^i coordinates of which are constant in time). Since $(\partial X'^A/\partial t)_x$ gives the components of this velocity in the X' system, it follows from (2.15) that $(\partial x^i/\partial t)_{X'}$ is the negative of this same velocity in the x frame. Introducing the notation

$$\left(\frac{\partial x^i}{\partial t}\right)_{X'} = -p^i, \tag{2.16}$$

then the combination of (2.13) and (2.14) yields

$$v^i = q^i + p^i, \tag{2.17}$$

a well-known result signifying that the total velocity of P' may be expressed as the sum of its velocity relative to the x frame and the velocity of the x frame itself or more precisely, the velocity of the point in the x frame with which P' is instantaneously in contact.

In the remainder of this section, we dispose of certain results preliminary to our main task. Selecting the set (x^i, t) as independent, then with the aid of (2.16),

$$\left(\frac{\partial \mathbf{g}_{,i}}{\partial t}\right)_x = \mathbf{p}_{,i} = p^k \,|_i\, \mathbf{g}_k , \tag{2.18}$$

and by application of (2.5b) to $X^A_{,i}$, and then to X^A we deduce

$$\left(\frac{\partial X^A_{,i}}{\partial t}\right)_X = -X^A_{,j}q^j_{,i} \ . \tag{2.19}$$

Recalling also that the Jacobian of transformation from x to X is given by [10, p. 63],

$$J = | T^A_B | = |\delta^A{}_B + U^A|_B| \ , \tag{2.20}$$

then since $(\partial T^A_B/\partial t)_X = (\partial U^A|_B/\partial t)_X$, by (2.1), (2.12), and (2.10) we arrive at

$$\left(\frac{\partial J}{\partial t}\right)_X = \frac{\partial J}{\partial T^A_B}\left(\frac{\partial T^A_B}{\partial t}\right)_X = J(T^{-1})^B_A V^A \ |_B \ . \tag{2.21}$$

Therefore, if in (2.8b) we replace $H^A|_B$ by $V^A|_B$ and introduce the result into (2.21) we obtain the identity

$$\left(\frac{\partial J}{\partial t}\right)_X = Jv^i \ |_i \tag{2.22}$$

which will be utilized in Sec. 3.

3. The time derivative of tensors. For the sake of clarity, we first consider the time derivative of a class of tensors with transformation connections of the type r^i_A , and confine attention, without loss in generality, to a mixed second order tensor which transforms according to

$$H^A_B = r^A_i r^j_B h^i_j \ , \tag{3.1}$$

and which may be regarded to include the transformation law appropriate to velocity and displacement components (Sec. 2).

Since there is no difficulty in defining the time derivative of H^A_B in the X frame, i.e. $(\partial H^A_B/\partial t)_X$, the main question confronting us is the definition of the corresponding quantity in the x frame or equivalently the establishment of an appropriate expression for the time derivative (or the time rate) of h^i_j which will be designated by Dh^i_j/Dt. Here this is achieved by seeking the tensor transformation of $(\partial H^A_B/\partial t)_X$ in the sense of the original transformation (3.1), i.e., Dh^i_j/Dt is the desired time derivative provided it transforms according to

$$\frac{Dh^i_j}{Dt} = r^i_A r^B_j \left(\frac{\partial H^A_B}{\partial t}\right)_X \ . \tag{3.2}$$

To this end, we apply $(\partial/\partial t)_X$ to (3.1), multiply throughout by $r^k_A r^B_l$, and by virtue of (3.2) write the resulting expression in the form

$$\frac{Dh^k_l}{Dt} = \left(\frac{\partial h^k_l}{\partial t}\right)_X + r^k_A\left(\frac{\partial r^A_i}{\partial t}\right)_X h^i_l - r^j_B\left(\frac{\partial r^B_l}{\partial t}\right)_X h^k_j \ . \tag{3.3}$$

In (3.3), the coefficient of terms involving h^i_l may be expressed as

$$\begin{aligned} r^k_A\left(\frac{\partial r^A_i}{\partial t}\right)_X &= r^k_A\left[\frac{\partial}{\partial t}\,(\mathbf{G}^A\cdot\mathbf{g}_i)\right]_X \\ &= \mathbf{g}^k\cdot\left(\frac{\partial \mathbf{g}_i}{\partial t}\right)_X \ , \end{aligned} \tag{3.4}$$

and since by (2.5b), (2.14), and (2.18),

$$\left(\frac{\partial \mathbf{g}_i}{\partial t}\right)_X = \mathbf{g}_{i,j} q^j + p^j \mid_i \mathbf{g}_j , \tag{3.5}$$

(3.4) becomes

$$r_A^k \left(\frac{\partial r_i^A}{\partial t}\right)_X = \left\{ {k \atop i\ \ j} \right\} q^j + p^k \mid_i . \tag{3.6}$$

Motivated by the fact that the Christoffel symbol of the second kind has the form

$$\left\{ {i \atop j\ \ k} \right\} = \mathbf{g}^i \cdot \mathbf{g}_{j,k} ,$$

we introduce the notation

$$\binom{i}{j} = \mathbf{g}^i \cdot \left(\frac{\partial \mathbf{g}_j}{\partial t}\right)_X = \left\{ {i \atop j\ \ k} \right\} q^k + p^i \mid_j , \tag{3.7}$$

and call $\binom{i}{j}$ the first time symbol. Substitution of (3.7) into (3.3) finally yields the desired result, i.e.,

$$\frac{Dh_l^k}{Dt} = \left(\frac{\partial h_l^k}{\partial t}\right)_X + \binom{k}{i} h_l^i - \binom{j}{l} h_j^k , \tag{3.8}$$

which is reminiscent of that for covariant derivative of second order tensors.

A generalization of the foregoing result is necessary, as not all tensors arising in mechanics of continua (especially in the constitutive equations) have transformation laws of the type (3.1). Notable among these are the stress σ^{ij} and the strain e^{ij} which have transformations

$$S^{AB} = J X^A_{,i} X^B_{,j} \sigma^{ij}, \tag{3.9}$$

$$E_{AB} = x^i_{,A} x^j_{,B} e_{ij} , \tag{3.10}$$

S^{AB} and E^{AB} being, respectively, the Kirchhoff tensor[3] and a measure of strain in the X frame. In order to distinguish between transformation connections of the type r_A^i and $x_{,A}^i$, temporarily we use Latin indices for the former and Greek indices for the latter. For our present purposes, it is sufficient to consider a mixed tensor of weight n which transforms according to the law

$$H^{A\Gamma}_{B\Delta} = J^n r^A_i r^j_B X^\Gamma_{,\alpha} x^\beta_{,\Delta} h^{i\alpha}_{j\beta} . \tag{3.11}$$

To establish the time derivative of $h^{i\alpha}_{j\beta}$, i.e. $Dh^{i\alpha}_{j\beta}/Dt$, as the tensor transformation of $(\partial H^{A\Gamma}_{B\Delta}/\partial t)_X$ in the sense of (3.11), the procedure is the same as in the case of (3.8), except that in addition to terms involving $\binom{i}{j}$ there also arise expressions $x^\alpha_{,\Lambda}(\partial X^\Lambda_{,\beta}/\partial t)_X$ and $(\partial J/\partial t)_X$ for which use may be made of (2.19) and (2.22), respectively. In addition it is expedient to define the second time symbol by

$$\begin{bmatrix} \alpha \\ \beta \end{bmatrix} = x^\alpha_{,\Lambda} \left(\frac{\partial X^\Lambda_{,\beta}}{\partial t}\right)_X = \binom{\alpha}{\beta} - v^\alpha \mid_\beta = -q^\alpha_{,\beta} , \tag{3.12}$$

where (2.17) and (3.7) have been used.

[3]In general, the stress tensor in the initial X frame has different representations. Here S^{AB} is a symmetric tensor.

Thus, omitting the details, the result sought for the time derivative of a tensor with the transformation law (3.11) is given by

$$\frac{Dh^{i\alpha}_{j\beta}}{Dt} = J^{-n} r^{i}_{A} r^{B}_{;j} x^{\alpha}_{,\Gamma} X^{\Delta}_{,\beta} \left(\frac{\partial H^{A\Gamma}_{B\Delta}}{\partial t} \right)_X = \left(\frac{\partial h^{i\alpha}_{j\beta}}{\partial t} \right)_X$$
$$+ \begin{pmatrix} i \\ k \end{pmatrix} h^{k\alpha}_{j\beta} - \begin{pmatrix} k \\ j \end{pmatrix} h^{i\alpha}_{k\beta} + \begin{bmatrix} \alpha \\ \mu \end{bmatrix} h^{i\mu}_{j\beta} - \begin{bmatrix} \mu \\ \beta \end{bmatrix} h^{i\alpha}_{j\mu} + n h^{i\alpha}_{j\beta} v^{k} |_{k} , \tag{3.13}$$

which includes (3.8) as a special case. With reference to the role played by the two time symbols $\begin{pmatrix} i \\ j \end{pmatrix}$ and $\begin{bmatrix} \alpha \\ \beta \end{bmatrix}$ in (3.13), it should be emphasized that the former is associated with the transformation connection r^{i}_{A} and the latter with $x^{\alpha}_{,\Gamma}$. Having distinguished between the two transformation connections by utilizing Latin and Greek indices, in the remainder of the paper we dispense with the latter and use only Latin indices.

Before proceeding further, we consider as an example, the application of (3.13) to the strain tensor

$$e_{ij} = \tfrac{1}{2}[u_i \,|_j + u_j \,|_i - u^m \,|_i \, u_m \,|_j]. \tag{3.14}$$

Since by (3.10) the strain tensor has only transformation connections of the type $x^{i}_{,A}$, then

$$\frac{De_{ij}}{Dt} = \left(\frac{\partial e_{ij}}{\partial t} \right)_X - \begin{bmatrix} k \\ i \end{bmatrix} e_{kj} - \begin{bmatrix} k \\ j \end{bmatrix} e_{ik} . \tag{3.15}$$

Applying (3.13) also to $u_i|_j$, [which by virtue of (2.7) transforms by both r^{i}_{A} and $x^{i}_{,B}$], and after computing $(\partial e_{ij}/\partial t)_X$ and substituting the results into (3.15), simplification yields

$$\frac{De_{ij}}{Dt} = \tfrac{1}{2}(v_i \,|_j + v_j \,|_i) \tag{3.16}$$

for the rate of strain tensor.

Although the application of (3.13) to the stress tensor will be postponed until the next section, it should be emphasized that the time derivative operator defined by (3.13) is applicable to any tensor, whether relative or absolute, provided the transformation law for the tensor in question is known *a priori*. This should not be construed to imply that the application of (3.13) inherently provides for any special requirements which may be demanded of the time derivative of a particular tensor or of the constitutive equations containing it. Indeed, such requirements depend on the transformation connections of the tensor which may even have more than one transformation law; this is exemplified by the representation of stress in the X frame which, in addition to S^{AB}, may also be defined by

$$\Sigma^{AB} = J r^{A}_{i} X^{B}_{,j} \sigma^{ij}. \tag{3.17}$$

4. The stress rate and comparisons with previous works. As pointed out in [2, 3, 5] and again in [7], an acceptable definition of stress rate must be one such that if the medium executes rigid motion only and if the stress is independent of time when referred to a coordinate system fixed in the medium (i.e., a coordinate system participating in the motion), then the stress rate must vanish when referred to this coordinate system.

It can be easily verified that the time derivative defined by (3.13) automatically satisfies the above criterion when applied to tensors which transform by $x^i_{,A}$ alone, as e.g., in the case of stress with the transformation law (3.9), i.e.

$$\frac{D\sigma^{ij}}{Dt} = \left(\frac{\partial\sigma^{ij}}{\partial t}\right)_X + \begin{bmatrix} i \\ k \end{bmatrix}\sigma^{kj} + \begin{bmatrix} j \\ k \end{bmatrix}\sigma^{ik} + \sigma^{ij}v^k\,|_k\,. \tag{4.1}$$

For, if we let x^i be the coordinate system participating in the rigid motion of the medium, then by (2.14), $q^i = 0$, and (2.17) and (3.12) become

$$v^i = p^i, \qquad \begin{bmatrix} i \\ j \end{bmatrix} = 0, \tag{4.2}$$

respectively. Furthermore, since for a rigid motion $v^k|_k = 0$, the last term in (3.13) vanishes even for relative tensors. Hence, it follows at once from (2.5b) and (4.1) that if ϕ is a representative of a tensor field (such as σ^{ij}) which transforms by $x^i_{,A}$ alone, then

$$\frac{D\phi}{Dt} = \left(\frac{\partial\phi}{\partial t}\right)_x \tag{4.3}$$

which meets the foregoing criterion, stated as condition (a) in Sec. 1.

Moreover, if the rate of stress invariants (such as $\sigma^{ij}\,\sigma_{ij}$) which arise in the theory of plasticity, is defined by application of (3.13), e.g.,

$$\frac{D}{Dt}\,(\sigma^{ij}\sigma_{ij}) = 2\sigma^{ij}\,\frac{D\sigma_{ij}}{Dt} \tag{4.4}$$

then, clearly, the vanishing of the stress rate will render the left-hand side of (4.4) stationary, and Prager's requirement [condition (b) in Sec. 1] is fulfilled.

If, instead of (3.9), the transformation law (3.17) is adopted for stress, then the application of (3.13) will not yield (4.1). To scrutinize this seemingly paradoxical result, we first observe that by combination of (3.9) and (3.17)

$$\textstyle\sum^{AB} = T^A_C S^{CB}, \tag{4.5}$$

and that

$$JX^A_{,i}X^B_{,j}\,\frac{D\sigma^{ij}}{Dt} = \left(\frac{\partial S^{AB}}{\partial t}\right)_X \tag{4.6a}$$

$$Jr^A_i x^B_{,j}\,\frac{D\sigma^{ij}}{Dt} = \left(\frac{\partial \sum^{AB}}{\partial t}\right)_X. \tag{4.6b}$$

Next, in conformity with the requirement demanded of a suitable definition of stress rate, we assume at the outset that $(\partial\sigma^{ij}/\partial t)_x = 0$ in the spatial coordinate system (participating in the rigid motion of the medium) for which (4.2) holds, and proceed to determine the partial derivative with respect to t of both $\sum^{AB}$ and S^{AB}. Through consideration of the stress vector and its transformation relation involving the stress tensor and the use of (2.5a), it may be shown that

$$\left(\frac{\partial \sum^{AB}}{\partial t}\right)_X = r^A_i\begin{Bmatrix} i \\ k \end{Bmatrix} r^k_C \textstyle\sum^{CB}, \tag{4.7}$$

with the aid of which application of $(\partial/\partial t)_X$ to (4.5) yields

$$\left(\frac{\partial S^{AB}}{\partial t}\right)_X = 0. \tag{4.8}$$

Hence, by the results (4.7) and (4.8), it is clear that *of the two representations for stress in the X frame, only* S^{AB} *with the transformation law* (3.9) *supplies a suitable stress rate*[4]. The time derivative of stress with the transformation law (3.17) specifically leads to

$$\frac{D\sigma^{ij}}{Dt} = \binom{i}{k}\sigma^{kj}$$

which jibes with (4.6b) and (4.7).

In the remainder of this section, we consider the various definitions of stress rate proposed in the recent literature all of which can be derived as special cases of (4.1). In this connection, it is pertinent to mention that since the stress rate (4.1) meets both conditions (a) and (b) discussed above, it follows that any definition of stress rate deduced from (4.1) will also satisfy these conditions.

(a) *Truesdell's definition.* If, as in [2], we select the x frame as one fixed in space, then $p^i = 0$, $q^i = v^i$, and by (3.12)

$$\begin{bmatrix} i \\ k \end{bmatrix} = \left\{ {i \atop k \quad j} \right\} v^j - v^i \,|_k \, . \tag{4.9}$$

With the aid of (2.5a) and (4.9), (4.1) reduces to

$$\frac{D\sigma^{ij}}{Dt} = \left(\frac{\partial \sigma^{ij}}{\partial t}\right)_x + \sigma^{ij}\,|_k\, v^k - v^i\,|_k\, \sigma^{kj} - v^j\,|_k\, \sigma^{ik} + \sigma^{ij} v^k\,|_k \, . \tag{4.10}$$

Observing that the first two terms on the right-hand side of (4.10) form the material derivative, designated in [2] by a superposed dot, then it becomes immediately apparent that (4.10) is identical with Truesdell's Eq. (1.10) in [2].

(b) *Jaumann's definition.* This definition [8], which has been rediscovered by Noll [3][5], may be deduced as a special case of (4.1) by considering the continuum to undergo rigid motion and referring the rate of stress to a coordinate system fixed in space. Under these conditions by (2.22), $p^i = 0$, and $q^i = v^i$ with

$$v_i\,|_j = \omega_{ij}\, , v^k\,|_k = g^{ki} v_i\,|_k = 0, \tag{4.11}$$

$\omega_{ij} = \frac{1}{2}(v_i|_j - v_j|_i)$ being the vorticity. With the aid of (2.5a), (3.12) and (4.11), (4.1) reduces to

$$\frac{D\sigma^i_j}{Dt} = \left(\frac{\partial \sigma^i_j}{\partial t}\right)_x + \sigma^i_j\,|_k\, v^k - \omega^{ik}\sigma_{kj} + \sigma^{ik}\omega_{kj}\, , \tag{4.12}$$

the result given by Noll [3].

(c) *Oldroyd's definition.* In order to establish the relationship between (3.13) and the corresponding expression in Oldroyd's work [4], it is convenient to consider separately tensor fields which transform by $x^i_{,A}$ and r^i_A. For this purpose, we limit ourselves (without loss in generality) to

$$\sigma^i_j = x^i_{,A} X^B_{,j} S^A_B\, , \tag{4.13}$$

and

$$h^i_j = r^i_A r^B_j H^A_B\, , \tag{4.14}$$

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[4]It is of interest to note that the transformation law for S^{AB} is of the same type as that for strain given by (3.10). This is in keeping with the remark made in Sec. 1 that the choices of stress, strain and their rates are intimately tied in with the constitutive equations of the media.

[5]In this connection, see the remarks made in [7] where other references employing Jaumann's definition are cited.

which have, respectively, the representations

$$S^A_B = X^{\ A}_{,i} x^i_{,B} \sigma^i_j \tag{4.15}$$

and

$$H^A_B = X^{\ A}_{,i} x^i_{,B} h^i_j \ , \tag{4.16}$$

in the X coordinate system.

Oldroyd's approach in [4] is to obtain the transformation into the X' system of $[\partial/\partial t\,(\ \ )]_x$ in the x frame; the latter frame (x) being taken as convected with the medium in which case $q^i = 0$, $p^i = v^i$, and by (3.7) and (3.12)

$$\binom{i}{j} = v^i\,|_j\ , \qquad \begin{bmatrix} i \\ j \end{bmatrix} = 0. \tag{4.17}$$

In order to show the reduction of (3.13) to Oldroyd's results for tensor fields with transformation connections of the type $x^i_{,A}$, we proceed to determine $D\sigma^i_j/Dt$ and DS'^A_B/Dt, which (when x^i is convected with the medium), are related through

$$\frac{DS'^A_B}{Dt} = X'^A_{,i} x^i_{,B'} \frac{D\sigma^i_j}{Dt}\ . \tag{4.18}$$

Taking into account (4.17), (3.13) when applied to σ^{ij} with the aid of (2.5b) yields

$$\frac{D\sigma^i_j}{Dt} = \left(\frac{\partial \sigma^i_j}{\partial t}\right)_X = \left(\frac{\partial \sigma^i_j}{\partial t}\right)_x\ . \tag{4.19}$$

The definition (3.13) may also be used to calculate (DS'^A_B/Dt) provided we identify x^i in (3.13) with X'^A as utilized in this section. Denoting by V'^A, Q'^A, and P'^A the counterparts[6] of v^i, q^i, and p^i, respectively, in the X' system which is fixed in space, it follows that $P'^A = 0$, $Q'^A = V'^A$, and by (3.7) and (3.12) when x^i is identified with X'^A

$$\begin{aligned} \binom{A}{B}' &= \begin{Bmatrix} & A & \\ B & & C \end{Bmatrix}' V'^C \\ \begin{bmatrix} A \\ B \end{bmatrix}' &= \begin{Bmatrix} & A & \\ B & & C \end{Bmatrix}' V'^C - V'^A\,|_{B'}\ . \end{aligned} \tag{4.20}$$

Applying (3.13) to the combination of (3.7) and (4.15), and utilizing (4.20), (2.5a), and (4.19), we obtain

$$\left(\frac{\partial \sigma^i_j}{\partial t}\right)_x X'^A_{,i} x^i_{,B'} = \left(\frac{\partial S'^A_B}{\partial t}\right)_{X'} + S'^A_B\,|_{C'}\, V'^C + V'^A\,|_{D'}\, S'^D_B + V'^D\,|_{B'}\, S'^A_D\ . \tag{4.21}$$

The right-hand side of (4.21) is Oldroyd's definition of stress rate which in [4] is denoted by $(\delta S'^A_B/\delta t)$.

We now turn our attention to the reduction of (3.13) to Oldroyd's result when the tensor in question transforms by r^i_A. Following the procedure which led to (4.21), the

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[6]It should be kept in mind that P'^A and Q'^A defined as counterparts of p^i and q^i in the X' system, are not the transformations $(X'^A_{,i}\ p^i)$ and $(X'^A_{,i}\ q^i)$ which hold only if X' and x are two different types of coordinate systems; in Sec. 4 of this paper these two coordinate systems are of the same type.

time derivatives of (4.14) and (4.16) with the aid of (4.17), (4.20), and (2.5a) may be written, respectively, as

$$\frac{Dh_j^i}{Dt} = \left(\frac{\partial h_j^i}{\partial t}\right)_x + v^i\,|_k\, h_j^k - v^k\,|_j\, h_k^i \tag{4.22}$$

and

$$\frac{DH_B'^A}{Dt} = \left(\frac{\partial H_B'^A}{\partial t}\right)_{X'} + H_B'^A\,|_{C'}\, V'^C, \tag{4.23}$$

which may be shown to be related through

$$\frac{Dh_j^i}{Dt} X'^A_{,i} x^j_{,B'} = \frac{DH_B'^A}{Dt}. \tag{4.24}$$

Substitution of (4.22) and (4.23) into (4.24) leads to an equation which when put in the form

$$\left(\frac{\partial h_j^i}{\partial t}\right)_x X'^A_{,i} x^j_{,B'} = \left(\frac{\partial H_B^A}{\partial t}\right)_{X'} + H_B'^A\,|_{C'}\, V'^C - V'^A\,|_{D'}\, H_B'^D + V'^D\,|_{B'}\, H_D'^A \tag{4.25}$$

is similar to (4.21) and agrees with the corresponding result in [4].

In identifying the prediction of (3.13) with Oldroyd's results, no mention was made of the time derivative of relative tensors included in [4]. If the transformation between x and X involves weighting, then a term of the type $v^k|_k$ will occur on both sides of (4.21) and (4.25) and will cancel each other. If the weighting is present in the transformation between x and X', then an additional term, obtained from the determinant $|X'^A_{,i}|$, will be present on the right hand side of (4.21) and (4.25) and these will again be in agreement with those in [4] for relative tensors.

(d) *Sedov's results.* The results given in [9] are special cases of (3.13) in two respects: first, as in [4], the coordinate system is convected; and second, in view of the method of approach, the treatment is limited to tensors which transform by r_A^i alone. On account of the latter, only the first time symbols arise which as in (c) above for convected coordinates are given by (4.17) and (4.20).

Since the vanishing of the time derivative (D/Dt) of the base vectors together with (4.17) and (4.20) yield expressions of the type $(\partial \mathbf{g}_i/\partial t)_x = v^k|_i \mathbf{g}_k$, then Sedov's main results expressed in Eqs. (9) of Ref. [9] can be obtained by direct application of (3.13). Thus, for example if h^{ij} transforms by r_A^i alone, we have

$$\frac{Dh^{ij}}{Dt} = \left(\frac{\partial h^{ij}}{\partial t}\right)_x + v^i\,|_k\, h^{kj} + v^j\,|_k\, h^{ik} \tag{4.26a}$$

$$\frac{DH'^{AB}}{Dt} = \left(\frac{\partial H'^{AB}}{\partial t}\right)_x + \begin{Bmatrix} A \\ C \quad D\end{Bmatrix}' V'^D H'^{CB} + \begin{Bmatrix} B \\ C \quad D\end{Bmatrix}' V'^D H'^{AC} \tag{4.26b}$$

which are, respectively, the third and fourth Eqs. (9) in Ref. [9]. The first and second of Sedov's Eqs. (9) may be obtained in a similar manner, while in order to deduce the last of Eqs. (9) in [9], we assume as Sedov has that the coordinate system x^i is Cartesian in which case by (3.7),

$$\begin{pmatrix} i \\ j \end{pmatrix} = \mathbf{g}^i \cdot \boldsymbol{\omega} \times \mathbf{g}_j = \omega_j^i\ .$$

The remaining results in [9] are merely specializations of the above and are derived with the further assumption that the coordinate systems x^i and X^A coincide at some particular instant of time.

Before closing this section it should be mentioned that (3.13) also may be shown to reduce to other definitions of stress rate, e.g., that given by Thomas [5], where a special Cartesian coordinate system called "kinematically preferred" is employed. As this reduction is reasonably lengthy, it will not be included here.

5. A coordinate derivative analogous to the time derivative (3.13). While the order of covariant derivatives of a tensor and $[\partial(\ )/\partial t]_X$ is commutative in a material coordinate system, such as interchangeability of order of derivatives, though often useful, is not in general permissible. With the primary aim of providing a mechanism for commutation in the x frame of $D(\ )/Dt$ and an appropriate coordinate derivative, we deduce in this section an expression for the latter which will be designated by double strokes (||).

As in Sec. 3, where the time derivative (3.13) is obtained as the tensor transformation of the total time derivative in the X frame, the coordinate derivative that we seek plays a similar role with respect to the covariant derivative, and again depends on the fact that not all tensors transform by the same law from X^A to x^i. For our present purposes, it will suffice to consider separately the following two transformation laws, i.e.,

$$h^i = r_A^{\cdot i} H^A, \tag{5.1}$$

and

$$\sigma^i = x^{\ i}_{,A} S^A. \tag{5.2}$$

We first take the partial derivative of (5.1) and with the use of (2.9) write $h^i_{,j}$ as

$$h^i_{,j} = \mathbf{g}^i_{,j} \cdot \mathbf{G}_A H^A + \mathbf{g}^i \cdot \mathbf{G}_{A,B} X^{\ B}_{,j} H^A + r_A^{\cdot i} H^A_{,B} X^{\ B}_{,j}\ . \tag{5.3}$$

Recognizing that the left-hand side of this equation and the first term on the right-hand side (with $\mathbf{G}_A H^A = \mathbf{g}_k h^k$), upon transfer, is the covariant derivative of h^i and combining the remaining two terms in the form

$$r_A^{\cdot i} X^{\ B}_{,j}\left[H^A_{,B} + \begin{Bmatrix} A \\ C \quad B\end{Bmatrix} H^C\right],$$

we arrive at

$$h^i\,|_j = r_A^{\cdot i} X^{\ B}_{,j} H^A\,|_B\ , \tag{5.4}$$

as the tensor transformation of the covariant derivative of H^A. But, more than this, the transformation law (5.4) also states that the derivative part of $H^A|_B$ transforms by $X^{\ B}_{,j}$ and not by r^B_j .

Before proceeding further, we recall [10] the vector sets

$$\begin{aligned} \mathbf{c}_i &= X^{\ A}_{,i} \mathbf{G}_A = R^k_i \mathbf{g}_k \\ \mathbf{c}^i &= x^{\ i}_{,A} \mathbf{G}^A = (R^{-1})^i_k \mathbf{g}^k \end{aligned} \tag{5.5}$$

and the Cauchy-Green measure of deformation

$$\begin{aligned} c^{ij} &= \mathbf{c}^i \cdot \mathbf{c}^j = x^{\ i}_{,A} x^{\ j}_{,B} G^{AB}, \\ c_{ij} &= \mathbf{c}_i \cdot \mathbf{c}_j = X^{\ A}_{,i} X^{\ B}_{,j} G_{AB}\ , \end{aligned} \tag{5.6}$$

where (2.10) and (2.6) have been employed on the right-hand sides of (5.5). We now turn to (5.2), take its partial derivative, use (5.5) and rearrange the result in a manner similar to (5.4) to obtain

$$\sigma^i_{,j} + \mathbf{c}^i \cdot \mathbf{c}_{k,j}\sigma^k = x^i_{,A}X^B_{,j}S^A\,|_B\,. \tag{5.7}$$

In keeping with the original transformation law (5.2), the left-hand side of (5.7) is the tensor transformation of the covariant derivative $S^A|_B$, where in particular we note that its derivative part transforms by $X^B_{,j}$.

Introducing the notation

$$\left[\left[{}_k{}^i{}_j\right]\right] = \mathbf{c}^i \cdot \mathbf{c}_{k,j}\,, \tag{5.8}$$

then the left-hand side of (5.7) may be designated by

$$\sigma^i\,||_j = \sigma^i_{,j} + \left[\left[{}_k{}^i{}_j\right]\right]\sigma^k \tag{5.9}$$

which is the appropriate tensor transformation of $S^A|_B$ in the X frame. Since the coordinate derivative (5.9) reduces to the covariant derivative in the X frame, to be consistent, we employ throughout this section the notation $\psi^i||_j$ for the coordinate derivative of any tensor regardless of its transformation law. In view of (5.1) and (5.4), as well as (5.2) and (5.7), it is understood that this coordinate derivative (1) is identical with the covariant derivative $\psi^i|_j$ if Ψ^A transforms by $r^A_{i.}$, and (2) it is defined by the right-hand side of (5.9) with σ^i replaced by ψ^i, if Ψ^A transforms by $X^A_{,i}$.

In order to generalize (5.9) for the coordinate derivative of any tensor, we need to establish the relationship between $\left[\left[{}_j{}^i{}_k\right]\right]$ and the Christoffel symbol $\left\{{}_j{}^i{}_k\right\}$. To this end, it will suffice to consider the transformation

$$u^i = r^i_A U^A,$$

the first coordinate derivative of which is

$$u^i\,||_j \equiv u^i\,|_j = r^i_A X^B_{,j} U^A\,|_B\,. \tag{5.10}$$

In contrast to the covariant derivative $u^i|_{jk}$, since in (5.10), one index transforms by r^i_A and the other by $X^B_{,j}$, the coordinate derivative $u^i||_{jk}$ reads as

$$\begin{aligned} u^i\,||_{jk} &\equiv u^i\,|_j\,||_k \\ &= u^i\,||_{j,k} + \left\{{}_l{}^i{}_k\right\}u^l\,||_j - \left[\left[{}_j{}^l{}_k\right]\right]u^i\,||_l \\ &= r^i_A X^B_{,j} X^C_{,k} U^A\,|_{BC}\,, \end{aligned} \tag{5.11}$$

which is obtained by direct computation.

By (5.5) and (5.8), as well as (2.9),

$$\begin{aligned} \left[\left[{}_j{}^i{}_k\right]\right] &= (R^{-1})^i_m\left[R^m_l\left\{{}_j{}^l{}_k\right\} - u^m\,|_{jk}\right] \\ &= \left\{{}_j{}^i{}_k\right\} - (R^{-1})^i_m u^m\,|_{jk}\,. \end{aligned} \tag{5.12}$$

On the other hand, if we solve for $\left[\left[{}_{j}{}^{i}{}_{k}\right]\right]$ from (5.11) with the aid of (2.9), we obtain

$$R^{m}_{\cdot i}\left[\left[{}_{j}{}^{i}{}_{k}\right]\right] = \left\{{}_{j}{}^{m}{}_{k}\right\} - \left\{{}_{l}{}^{m}{}_{k}\right\}u^{l}\,|_{j} - u^{m}\,||_{jk} + \left\{{}_{l}{}^{m}{}_{k}\right\}u^{l}\,|_{j} - \left[\left[{}_{j}{}^{l}{}_{k}\right]\right]u^{m}\,|_{l}\,. \tag{5.13}$$

Combination of (5.12) and (5.13), following a lengthy manipulation, yields the relationship between the two symbols $\left[\left[{}_{j}{}^{i}{}_{k}\right]\right]$ and $\left\{{}_{j}{}^{i}{}_{k}\right\}$ in the form

$$\left[\left[{}_{j}{}^{i}{}_{k}\right]\right] = -u^{i}\,||_{jk} + \left\{{}_{j}{}^{i}{}_{k}\right\} \tag{5.14}$$

which is similar to the difference between the two time symbols given by (3.12).

The relation (5.14) may be used to advantage in obtaining the partial derivative of $J = |\,T^{A}_{B}|$. Thus, with the aid of (2.10) and (5.14),

$$J_{,k} = Ju^{m}\,||_{mk}\,, \tag{5.15}$$

an expression similar to (2.22).

With (5.4) and (5.7) as well as the result (5.15) at our disposal, the generalization of the coordinate derivative for any tensor field is immediate. Again, as in Sec. 3, employing temporarily Latin indices for transformation by r^{A}_{i} and Greek indices for transformation by $X^{A}_{,i}$, if $H^{A\Gamma}_{B\Delta}$ transforms according to

$$J^{n}h^{i\alpha}_{j\beta} = r^{i}_{A}r^{B}_{j}\,{}^{\alpha}_{,\Gamma}\,X^{\Delta}_{,\beta}H^{A}_{\Delta}\,, \tag{5.16}$$

then it can be shown as in the case of (5.4) and (5.7) that the desired coordinate derivative has the form

$$\begin{aligned} h^{i\alpha}_{j\beta}\,||_{k} = {} & h^{i\alpha}_{j\beta,k} \\ & + \left\{{}_{m}{}^{i}{}_{k}\right\}h^{m\alpha}_{j\beta} - \left\{{}_{j}{}^{m}{}_{k}\right\}h^{i\alpha}_{m\beta} \\ & + \left[\left[{}_{\mu}{}^{\alpha}{}_{k}\right]\right]h^{i\mu}_{j\beta} - \left[\left[{}_{\beta}{}^{\mu}{}_{k}\right]\right]h^{i\alpha}_{j\mu} \\ & + nh^{i\alpha}_{j\beta}u^{m}\,||_{mk}\,. \end{aligned} \tag{5.17}$$

Dispensing again with the use of Greek indices, and considering the transformations of

$$J^{n}h^{ik}_{jl}\,||_{m} \quad \text{and} \quad J^{n}\frac{Dh^{ik}_{jl}}{Dt}\,,$$

the truth of

$$\left(\frac{Dh^{ik}_{jl}}{Dt}\right)||_{m} = \frac{D}{Dt}\left(h^{ik}_{jl}\,||_{m}\right) \tag{5.18}$$

can be verified. That the identity (5.18) holds is not surprising if we recall that (3.13) and (5.17) were deduced, respectively, as the tensor transformations of $(\partial(\ \ )/\partial t)_X$ and $(\ \ )|_M$ which commute in the X frame.

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