main diagonal, then there exists a solution X of Eq. (1) for $-\infty < a \leq x \leq b < +\infty$ except at the zeros of the invariant factors of A.

Since the matrices of (2) are similar,

$$\begin{aligned} \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} &= \begin{bmatrix} PAP^{-1} & PCQ^{-1} \\ 0 & QBQ^{-1} \end{bmatrix} = M \\ \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} &= \begin{bmatrix} PAP^{-1} & 0 \\ 0 & QBQ^{-1} \end{bmatrix} = N, \end{aligned} \tag{4}$$

and the matrices M, N are similar. Form the equation

$$DU'' + UF = G, \tag{5}$$

where $D = PAP^{-1}$, $F = QBQ^{-1}$, $G = PCQ^{-1}$. Consider Eq. (5) element-wise, namely,

$$d_{ii}u''_{ij} + u_{ij}f_{jj} = g_{ij}\,, \qquad (i, j = 1, 2, \cdots, n), \tag{6}$$

where $u''_{ij} \in U''$, $u_{ij} \in U$, $d_{ii} \in D$, $f_{jj} \in F$, and $g_{ij} \in G$.

Next to show that Eq. (6) always has a solution. W. E. Roth* has shown that if the matrices of (2) are equivalent, then for the elements g_{ij} of G: (i) g_{ij}, for $1 \leq i \leq \alpha$, $1 \leq j \leq \beta$, is a multiple of the greatest common factor of d_{ii} and f_{jj}; (ii) g_{ij} for $1 \leq i \leq \alpha$, $\beta < j \leq n$, is a multiple of d_{ii}; (iii) g_{ij}, for $\alpha < i \leq n$, $1 \leq j \leq \beta$, is a multiple of f_{jj}; (iv) g_{ij}, for $\alpha < i \leq n$, $\beta < j \leq n$, is identically zero. Thus g_{ij} cannot be different from zero when both d_{ii}, f_{jj} are identically zero, and in each of the four cases above, (6) has a solution.

Let $U = P\,X\,Q^{-1}$, then $U'' = PX''Q^{-1} = (PXQ^{-1})''$, so Eq. (5) may be written as

$$(PAP^{-1})(PX''Q^{-1}) + (PXQ^{-1})(QBQ^{-1}) = PCQ^{-1}. \tag{7}$$

Multiplying (8) on the left by P^{-1} and on the right by Q, we have Eq. (1); thus $X = P^{-1}\,UQ$ is a solution of Eq. (1).

FURTHER PROPERTIES OF CERTAIN CLASSES OF TRANSFER FUNCTIONS: II**

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This note is a sequel to a previously published paper [1]. The notation and terminology used here is the same as before. The purpose of this note is to point out a consequence of some previously published results [1, 2], which is immediately applicable to rational transfer functions that have no poles in the right-half plane and have at least twice as many poles as zeros. Such transfer functions arise quite commonly in physical systems.

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*W. E. Roth, *The Equation $AX - YB = C$, and $AX - XB = C$ in Matrices*, Proc. Am. Math. Soc. **3**, 392–396 (1952).

**Received October 5, 1960.

Let $F(s)$ be such a transfer function. It may be written as follows, where $m \geq 1$.

$$F(s) = \frac{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_0}{s^m + b_{m-1}s^{m-1} + \cdots + b_0} = \frac{N(s)}{D(s)}. \tag{1}$$

The symbol $f(t)$ will denote the inverse Laplace transform of $F(s)$ that is zero for negative t. (This is the unit impulse response of the system.)

Theorem: Let $F(s)$ be given by (1) *and let $D(s)$ have no roots in the right-half s-plane. For m odd, let $n \leq (m - 1)/2$ and, for m even, let $n \leq m/2$. Then*

$$| f(t) | \leq \frac{| a_n | t^{m-n-1}}{(m - n - 1)!} + \frac{| a_{n-1} | t^{m-n}}{(m - n)!} + \cdots + \frac{| a_1 | t^{m-2}}{(m - 2)!} + \frac{| a_0 | t^{m-1}}{(m - 1)!}. \tag{2}$$

Proof. Let $D(s)$ be a Hurwitz polynomial and let $m \geq 2$, at first. By Theorem 3 of [2], $1/D(s)$ is a subclass m function. Hence, by Theorem 2 of [1], $s^\mu/D(s)$ is a class $(m - \mu)$ function where μ is any integer in the range, $0 \leq \mu \leq (m - 1)/2$, if m is odd and in the range, $0 \leq \mu \leq m/2$, if m is even. Under these restrictions on μ, the inverse Laplace transform of $s^\mu/D(s)$ is the μth derivative of the inverse Laplace transform of $1/D(s)$ Letting $\mathcal{L}^{-1}$ denote the inverse Laplace transform, Theorem 3 of [1] implies that

$$\left| \mathcal{L}^{-1}\left[\frac{s^\mu}{D(s)}\right] \right| \leq \frac{t^{m-n-1}}{(m - n - 1)!}.$$

Combining this result with (1), (2) is obtained.

If $m = 1$, the hypothesis implies that $m = 1$ and $n = 0$. In this case, (2) becomes $| f(t) | \leq | a_0 |$ and this result is easily established.

The proof is completed by noting that the conclusion still holds when the restriction that $D(s)$ is a Hurwitz polynomial is replaced by the weaker restriction that $D(s)$ has no roots in the right-half s-plane. This is so since the unit impulse response is a continuous function of the pole positions of the corresponding transfer function.

The principal advantage of this theorem over the previously published results is the following. In the previous cases one always had to establish whether a transfer function is a member of some class of functions before a particular bound could be placed on the corresponding unit impulse response. In this case one must establish that the degrees of $N(s)$ and $D(s)$ are appropriate and that $D(s)$ has no poles in the right-half s-plane. The restriction on m and n can be checked by inspection while in many practical cases it is known beforehand that $D(s)$ has no roots with positive, real parts. For instance, if the system under consideration is a lumped, linear, fixed, finite, and passive one, the transfer function will be rational and have no poles in the right-half s-plane.

References

1. A. H. Zemanian, *Further properties of certain classes of transfer functions*, Quart. Appl. Math. **18,** 223–228 (1960)
2. A. H. Zemanian, *On transfer functions and transients*, Quart. Appl. Math. **16**, 273–294 (1958)