— NOTES —

A NEW INTERPRETATION OF THE PLASTIC MINIMUM PRINCIPLES*

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In the theory of elasticity, the principle of minimum potential energy states that

$$\Pi \equiv \int_V \left(\int \sigma_{ij}\, d\epsilon_{ij} \right) dV - \int_{S_T} T_i u_i \, dS - \int_V F_i u_i \, dV \tag{1}$$

is a minimum for the actual state among all kinematically admissible states. A mathematically similar principle for elastic/plastic materials states that

$$W(\sigma_*' , \epsilon_*') - W(\sigma', \epsilon') \geq 0, \tag{2}$$

where the functional W is defined for any independent states of stress σ_{ij}^0 and strain ϵ_{ij}^* by

$$W(\sigma_0 , \epsilon_*) = \int_V \sigma_{ij}^0 \epsilon_{ij}^* \, dV - \int_{S_T} T_i^0 u_i^* \, dS - \int_V F_i^0 u_i^* \, dV \tag{3}$$

and primes denote differentiation with respect to t. We shall consistently use the asterisk to denote an arbitrary kinematically admissible state and unmarked symbols to denote the actual state. Precise definitions of terms, proofs of (2), and credits for original references may be found in any of several books such as [1, 2, 3, 4, 5].[1]

Despite the familiarity of (2), the principle has never received a name. This is doubtless due to the fact that (2) has the unwieldy dimensions of rate of power and it is difficult to find any physical significance for $W(\sigma', \epsilon')$. The purpose of the present note is to show that the principle embodied in (2) may be given either of the following two interpretations.

Theorem. Among all kinematically admissible rate states the actual rate state minimizes the time variations of

(*1*) *the potential dissipation function* $W(\sigma, \epsilon')$;
(*2*) *the pseudo-potential*[2] *energy* Π.

The proofs of these two assertions rest on the principle of virtual work and related principles involving rates. In the present notation

$$W(\sigma, \epsilon_*^{(t)}) = W(\sigma, \epsilon^{(t)}), \tag{4}$$

where the superscript (t) stands for any number of time differentiations. Also, we note that since the existing state is always known and only the rates are varied, only rate quantities need be distinguished by an asterisk. Therefore, we can write

$$\Pi_*' - \Pi' = W(\sigma, \epsilon_*') - W(\sigma, \epsilon') = 0 \tag{5}$$

and

$$\Pi_*'' - \Pi'' = W(\sigma, \epsilon_*'') - W(\sigma, \epsilon'') + W(\sigma_*', \epsilon_*') - W(\sigma', \epsilon') \geq 0. \tag{6}$$

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[1]Numbers in brackets refer to the list of references collected at the end of the paper.

[2]Since $\int \sigma_{ij} d\epsilon_{ij}$ may be partly dissipated for a plastic material, Π as defined by (1) is no longer a potential energy.

Equation (5) is an immediate consequence of (4), and the inequality (6) follows from (5) and (2). The first statement of the theorem is thus verified and the second statement becomes evident when $\Pi(t + \Delta t)$ is expanded in a Taylor's series in Δt about the time t.

If the complementary energy is defined by

$$\Pi_c \equiv \int_V \left(\int \epsilon_{ij} \, d\sigma_{ij} \right) dV - \int_{S_D} T_i u_i \tag{7}$$

and a functional W_c by

$$W_c(\sigma_0 , \epsilon_*) = \int_V \sigma^0_{ij} \epsilon^*_{ij} - \int_{S_D} T^0_i u^*_i \tag{8}$$

then the elastic principle of complementary energy states that Π_c is a minimum for the actual state among all statically admissible states and the analogous plastic principle states that

$$W_c(\sigma_0' , \epsilon_0') - W_c(\sigma', \epsilon') \geq 0. \tag{9}$$

Just as for the first principle one can easily prove two consequences of (9).

Theorem. Among all statically admissible rate states the actual rate state minimizes the time variations of

(*1*) *the complementary dissipation function* $W_c(\sigma', \epsilon)$:
(*2*) *the complementary energy* Π_c .

References

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ON THE SIMULTANEOUS DIAGONALIZATION OF TWO SEMI-DEFINITE MATRICES*

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1. Introduction. The use of congruency transformations for simultaneously diagonalizing two symmetric matrices, one of which is definite, is well known. One merely diagonalizes the definite matrix to (plus or minus) unity. This is then followed by an orthogonal transformation which diagonalizes the other matrix while preserving the unit matrix already obtained [1]. If, instead of being definite, one matrix is semi-definite,

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