ON A PROBLEM OF MINIMUM WEIGHT DESIGN*

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Summary. A problem of optimal design for perfectly plastic, isotropic structures is analyzed. It is shown that for such structures as plates or shells, an extremum of the volume, if it exists, is either a local maximum or a minimum.

1. Condition for an extremum of the volume. Consider a region R of space that is bounded by a regular surface S. On the part S_T of S, let the non-vanishing surface tractions T_i be prescribed, and on another part S_U of S, let the velocity u_i be required to vanish. It is assumed that S_T and S_u comprise the whole of the surface S. A rigid, perfectly plastic body B is to be designed to the following specifications: S_T and S_U form parts of the surface S_B of B; the remainder S_B' of S_B is to be free from surface tractions, while S_T is loaded in the prescribed manner and S_U is rigidly supported; the body B is to be contained in R and should reach its load carrying capacity under the prescribed surface tractions; it is to have the minimum volume possible under these conditions. This problem has been studied by Drucker and Shield [4, 5]†; we shall briefly discuss it to lay the foundation for subsequent work.

Consider a rigid, perfectly plastic body C that satisfies all conditions of the problem except that its volume V_c need not represent a minimum. As this body is supposed to reach its load carrying capacity under the prescribed surface tractions, there exists a stress field σ_{ij}^c that is statically admissible for these loads and does nowhere exceed the yield point. Moreover, there exists a kinematically admissible velocity field u_i^c the strain rates ϵ_{ij}^c of which are compatible with the stresses σ_{ij}^c and do not vanish identically. The principle of virtual velocities then furnishes the relation‡

$$\int \sigma_{ij}^c \epsilon_{ij}^c \, dV_c = \int T_i u_i^c \, dS_T . \tag{1}$$

Denoting the surface of this body by $S_c = S_T + S_u + S_c'$, we consider a modification of the stress-free part S_c' of this surface, such, that the altered body remains at the yield point under the same surface tractions T_i. If we move S_c' only outward (boundary E' in Fig. 1), the load-carrying capacity, in general, will increase (*cf.* [15]). So, the variation of S_c' should be performed in such a way that a part of S_c' is moved outside, the remainder inside of the body V_c (boundary E in Fig. 1). Further, we assume that there exists a stress field σ_{ij}^* defined throughout the volume $V^* = V_c + \delta V$ of the modified body that is statically admissible for the prescribed surface tractions and does nowhere exceed the yield limit. By the first fundamental theorem of limit analysis ([14], p. 40), it follows from this assumption that the prescribed surface tractions cannot exceed the load carrying capacity of the modified body. We further assume that the velocity field u_i^c is continued in a kinematically admissible manner into any volume that has been

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†Numbers in square brackets refer to the Bibliography at the end of the paper.
‡Body forces which do not alter the final conclusions, are not included here.

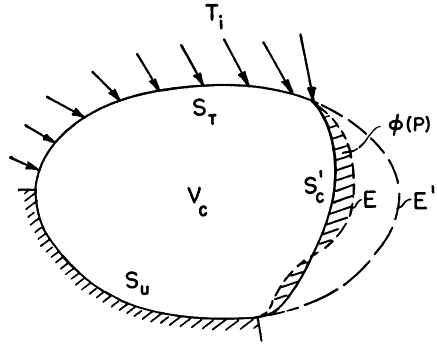

FIG. 1

added to V_c. It means that the field of strain rates ϵ_{ij}^c derived from u_i^c satisfies everywhere the incompressibility condition; moreover ϵ_{ij}^c are continuous functions within δV and on the surface S_c'. Obviously, in general, there are many such extensions beyond the volume V_c. If some additional kinematical constraints are imposed, the field of ϵ_{ij}^c has to be compatible with them and the number of such extensions may be reduced, even to one possible field. (e.g. plates, shells etc. subject to thickness variation). In the volume δV let σ_{ij}^c denote a stress field that corresponds to the extended strain rate field by the theory of the plastic potential ([14], p. 15) excluding rigid regions, where σ_{ij}^c is not defined. Setting $\sigma_{ij}^* = \sigma_{ij}^c + \delta\sigma_{ij}$ throughout the volume V^*, and applying the principle of virtual velocities to the stresses σ_{ij}^* and the strain rates ϵ_{ij}^c, we obtain

$$\int \sigma_{ij}^c \epsilon_{ij}^c \, dV_c + \int \sigma_{ij}^c \epsilon_{ij}^c \, d(\delta V) + \int \delta\sigma_{ij} \epsilon_{ij}^c \, dV^* = \int T_i u_i \, dS_T . \tag{2}$$

Here, the power of plastic dissipation $D = \sigma_{ij}^c \epsilon_{ij}^c$ is completely specified by the strain rates ϵ_{ij}^c ([14], p. 37).

Subtracting (1) from (2), we obtain:

$$\int D(\epsilon_{ij}^c) \, d(\delta V) = -\int \delta\sigma_{ij} \epsilon_{ij}^c \, dV^*. \tag{3}$$

The following theorem may now be proved:* *If* $0 < D(\epsilon_{ij}^c) = \bar{D} = \text{const.}$ *on* S_c' *and* $D(\epsilon_{ij}^c) \geq \bar{D}$ *throughout* V_c, *and if the velocity field* u_i^c *can be so continued in a kinematically admissible manner beyond* S_c' *that the strain rates of this continuation satisfy* $D(\epsilon_{ij}^c) \leq \bar{D}$, *the body C has the minimum volume that can be obtained if the given loads are not to exceed the load carrying capacity.* Indeed, setting $D(\epsilon_{ij}^c) = \bar{D} + \Delta D$, we write (3) in the form:

$$\bar{D} \, \delta V = -\int \delta\sigma_{ij} \epsilon_{ij}^c \, dV^* - \int \Delta \, D d(\delta V). \tag{4}$$

By the principle of the maximum specific power of dissipation ([14], p. 37), we have $\delta\sigma_{ij}\epsilon_{ij}^c \leq 0$, i.e. the first term on the right side of (4) is always positive and so is the second, since for points in the exterior of the body $\Delta D \leq 0$, $d(\delta V) > 0$, while $\Delta D \geq 0$, $d(\delta V) < 0$ for interior points. Thus, $\delta V > 0$ whenever S_c' is modified so that the pre-

*An analogous theorem has been stated in Ref. [5]; it is not clear however, which parts of the body were assumed to vary. If the part S_u may also undergo the variation, this theorem will no longer be true.

scribed surface tractions do not exceed the load carrying capacity of the modified body.

A proof of this theorem may also be provided by means of the second theorem of limit analysis*. Indeed, considering the modified body V^{**}, we assume that it has not yet reached or is just reaching its load carrying capacity. The kinematically admissible velocity field u_i^c cannot therefore be unstable ([14], p. 40) and we have

$$\int D(\epsilon_{ij}^c)\, dV^* \geq \int T_i u_i^c \, dS_T \,. \tag{5}$$

Setting $D = \bar{D} + \Delta D$ and comparing (1) and (5) one finds

$$\bar{D}\, \delta V \geq -\int \Delta\, Dd(\delta V) \geq 0, \tag{6}$$

which completes the proof.

A special case of this theorem arises when $D(\epsilon_{ij}^c) = \bar{D} =$ const. inside and outside of the body V_c. Then $\Delta D = 0$ and the right side of Eq. (4) contains only the first term. This case occurs, for instance, in discs, membranes or sandwich plates. If we consider a disc that is subjected to prescribed surface tractions T_i on all of the cylindrical surface S_T forming its "edge", only the lateral surfaces S_c', that is the thickness of the disc, may be varied to minimize the volume. Since stress and strain rate are constant across the thickness of the disc, the constancy of $\bar{D}$ on the lateral surfaces implies that $D(\epsilon_{ij}^c) = \bar{D} =$ const everywhere. This design criterion for discs has been given in [4] and [5].

Difficulties arise, however, when one deals with structures, such as plates and shells, for which $D(\epsilon_{ij}^c)$ is greater in the exterior than in the interior of V_c. The theorem proved above then is no longer applicable. Nevertheless, we shall try to choose the surface S_c' so as to render $D(\epsilon_{ij}^c) = \bar{D} =$ const. on S_c'. We shall see that this type of design has certain advantages.

An infinitesimal modification of the surface S_c' can be specified in terms of the normal displacement $\eta\varphi(P)$ of the typical point P. Here, η is an infinitesimal constant, and $\varphi(P)$ a continuous function of the position of the point P on S_c'. If $\delta\sigma_{ij}$ is of the first order* (i.e. of the same order as η), then $\delta\sigma_{ij}\epsilon_{ij}^c$ is negative and of the second or lower order according to whether the body V^* is everywhere at the yield point or not. Thus, the first term on the right side of Eq. (4) is positive and of the second or lower order, while the second term is negative because now $\Delta D \geq 0$, $d(\delta V) > 0$ in the exterior and $\Delta D \leq 0$, $d(\delta V) < 0$ in the interior of V_c. This second term is likewise of the second order, since ΔD is a first order quantity in the vicinity of S_c'. Thus, Eq. (4) implies the following: *a body for which* $D(\epsilon_{ij}^c) = \bar{D} =$ *const. on* S_c' *attains either a local minimum or a local maximum,* depending on the sign of the expression on the right side of Eq. (4). This sign must therefore be studied in each individual case. It should be noted that, whereas the minimum is not necessarily analytical, the maximum must be analytical.

It is interesting to note, that in the case considered last the second theorem of limit

*The author is indebted to Professor W. Prager for this remark. In fact, the proof based on limit analysis theorems has been presented in Ref. [5].

**The conditions that the function $\varphi(P)$ has to satisfy in order to ensure $\delta\sigma_{ij}$ to be of the first order are not yet known in the general case. In the following, this problem will be discussed in detail for a circular plate, symmetrically loaded.

analysis does not provide sufficient information on the sign of δV. Indeed, the inequality (6) now shows only that δV is greater than a negative quantity of the second order. From this fact, no conclusions can be drawn regarding the sign or order of δV. Equation (4), however, shows that we have an extremum of the volume.

A somewhat different formulation of the problem of optimum design arises for bodies that are required to have a plane, axis, or center, of symmetry. Then any modification of a part of the surface S_c' entails the corresponding modification of the symmetrically located part. If the prescribed surface tractions T_i and the kinematically admissible velocity field u_i^c need not have the symmetry properties of the body, the condition of constant $D(\epsilon_{ij}^c)$ on S_c' must be replaced by the condition

$$D^{(1)}(\epsilon_{ij}^c) + D^{(2)}(\epsilon_{ij}^c) = \bar{D} = \text{const.}, \tag{7}$$

where $D^{(1)}$ and $D^{(2)}$ denote the values of $D(\epsilon_{ij}^c)$ at a typical pair of symmetrically situated points of S_c'.

2. Design of plates. Consider a solid plate of given plane-form that is subjected to prescribed transverse loads and supported in a given manner. A particular design of this plate is specified by the distribution of the plate thickness over the plane-form, and the discussion will be restricted to designs that reach their load carrying capacity under the given loads. The problem of finding the design of minimum volume has been thoroughly investigated for circular plates under rotationally symmetric loading [1, 3, 6, 7] and some results have been obtained for other shapes or loads [2, 8, 9]. Drucker and Shield (4, 5) have established conditions under which the volume of a plate is stationary with respect to neighboring designs. As we shall see, these conditions may lead to a local maximum as well as a local minimum.

Let $2h_c$ be the variable thickness of a design for which the given loads represent the load carrying capacity. Denote the rates of extension and shear at the lower surface of the plate by ϵ_{ij}^0, $(i, j = 1, 2)$, and assume that the specific power of dissipation has the constant value $\bar{D}$ at this surface:

$$D(\epsilon_{ij}^0) = \bar{D}. \tag{8}$$

As is customary in plate theory, we shall assume the strain rates to vary linearly over the thickness of the plate. At the distance x_3 below the median plane of the plate, we therefore have the strain rates

$$\epsilon_{ij}^c = (x_3/h_c)\epsilon_{ij}^0, \qquad (i, j = 1, 2). \tag{9}$$

Since the dissipation function is homogeneous of the order one in the strain rates and does not change its value when the signs of all strain rate components are reversed, we have by (8) and (9)

$$D(\epsilon_{ij}^c) = (|\, x_3 \,|/h_c)\, \bar{D}. \tag{10}$$

Equation (9) also represents the kinematically admissible extension of the strain rate field beyond the lateral surfaces of the plate that will be used in the following.

The state of stress at a typical point of the fully plastic plate will be treated as plane, and the notation σ_{ij}^0, $(i, j = 1, 2)$, will be used for the stresses in the lower half of the plate, those in the upper half having the same intensities but opposite signs. The same stresses are compatible with the extension (9) of the strain rate field.

If the variation of the plate thickness is denoted by $2\delta h$, and the element of area of the median plane by dA, the terms on the right side of (4) can be evaluated as follows:

$$\begin{aligned}\int \delta\sigma_{ij}\epsilon^c_{ij}\, dV^* &= 2\int dA \int_0^{h_c+\delta h} \delta\sigma_{ij}\epsilon^0_{ij}(x_3/h_c)\, dx_3 \\ &= \int \delta\sigma_{ij}\epsilon^0_{ij}[(h_c + \delta h)^2/h_c]\, dA,\end{aligned} \tag{11}$$

$$\begin{aligned}\int \Delta\, Dd(\delta V) &= 2\int dA \int_{h_c}^{h_c+\delta h} [(x_3/h_c) - 1]\, \bar{D}\, dx_3 \\ &= \bar{D}\int (\delta h^2/h_c)\, dA.\end{aligned} \tag{12}$$

Equation (4) therefore furnishes

$$\delta V = -\int (\delta h^2/h_c)\, dA - 1/\bar{D}\int \delta\sigma_{ij}\epsilon^0_{ij}[(h_c + \delta h)^2/h_c]\, dA. \tag{13}$$

If the yield locus in the space with the rectangular Cartesian coordinates σ_{11} , σ_{22} , $\sigma_{12}(2)^{\frac{1}{2}}$ has continuously turning tangent plane, $\delta\sigma_{ij}\epsilon^0_{ij}$ is negative and of the second order. The two terms on the right side of (13) are therefore of the second order and of opposite signs. This indicates that the volume of the original plate is stationary in comparison to neighboring plates that reach their load carrying capacity under the prescribed surface tractions, but no general statement can be made regarding the character of this stationary value (maximum, minimum, or saddle point).

If, on the other hand, the yield locus has a plane face and the states of stress in the lower half of the plane are represented by interior points of this face, for the original as well as the modified plate, then $\delta\sigma_{ij}\epsilon^0_{ij} = 0$. The right-hand side of (13) therefore is negative and of the second order, so that the original plate represents an analytical local *maximum* of the volume. The same remark applies if the yield locus contains a straight segment and the states of stress in the lower half of the plate are represented by interior points of this segment for both the original and the modified plate.

Finally, if the state of stress throughout the lower half of the plate is represented by points of an edge of the yield locus, $\delta\epsilon_{ij}\epsilon^0_{ij}$ will as a rule be negative and of the first order, so that the first term on the right side of (13) can be neglected. The original plate then represents a non-analytical local minimum of the volume. This statement, however, must be modified, if for each considered stress point σ^0_{11} , σ^0_{22} , $\sigma^0_{12}(2)^{\frac{1}{2}}$ on an edge of the yield locus, the vector with the components ϵ^0_{11} , ϵ^0_{22} , $\epsilon^0_{12}(2)^{\frac{1}{2}}$ happens to be normal to one of the faces intersecting in this edge. If then the corresponding stress point for the modified plate lies on this face, $\delta\sigma_{ij}\epsilon_{ij}$ will be negative and of the second order; if, however, this stress point lies on the other face, $\delta\sigma_{ij}\epsilon_{ij}$ will be negative and of the first order. In the first case, we have a stationary value, in the second a local minimum of the volume.

Equation (13) may also be rewritten in the following form:

$$\delta V = -\int (\delta h^2/h_c)\, dA - \frac{1}{\bar{D}}\int \delta M'_{ij}\kappa^c_{ij}[(h_c + \delta h)/h_c]^2\, dA, \tag{14}$$

where $\delta M'_{ij} = \delta\sigma_{ij}h_c^2$ denotes the variation of bending moments due to the change of the stress components only and κ^c_{ij} is the rate of curvature. Figure 2 represents the

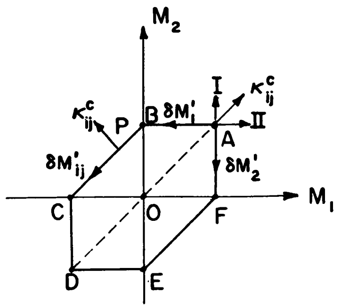

FIG. 2

Tresca yield condition referred to the principal bending moments. In view of the above, we conclude that for sides of the hexagon the optimal solution represents a maximum of the volume, while for the corners of the hexagon the plate will have either a minimal volume or its volume will correspond to a saddle point. The last case occurs, for instance, when κ_{ij}^c coincides with one of the normals to the adjacent sides, say I at the corner A; the solution obtained will then represent a maximal volume for all states AB and will be a minimum with respect to all states AF, i.e. it will correspond to a saddle point.

Example 1. Consider first a simply supported circular plate of the radius a under the uniformly distributed load q, and adopt the Tresca yield condition. On account of the rotational symmetry, the principal bending moments are the radial moment M_r and the circumferential moment M_φ ; the associated strain rates are the radial and circumferential rates of curvature κ_r and κ_φ .

Hopkins and Prager [1] treated this problem of optimum design assuming that the entire plate is at states of stress corresponding to the corner A of the hexagon (Fig. 2). For the plastic moment $M_0 = \sigma_0 h_c^2$ (σ_0 being yield stress in simple tension) they obtained the expressions

$$M_0 = \frac{q}{4}(a^2 - r^2), \qquad h_c = \tfrac{1}{2}(q/\sigma_0)^{1/2}(a^2 - r^2)^{1/2}, \qquad \text{(Fig. 3a)}. \tag{15}$$

The corresponding distribution of the rate of deflection w can be determined as follows. Starting from the condition $\bar{D} =$ const. we write

$$\bar{D} = \frac{M_r\kappa_r + M_\varphi\kappa_\varphi}{h_c} = \alpha, \quad \text{or} \quad \frac{1}{r}\frac{dw}{dr} + \frac{d^2w}{dr^2} = -\frac{\bar{\alpha}}{(a^2 - r^2)^{1/2}}, \tag{16}$$

where $\alpha =$ const. and $\bar{\alpha} = 2\alpha/(g\sigma)^{1/2}$. Integrating Eq. (16), one obtains

$$w = \bar{\alpha}\left[-a \ln\frac{r}{a} + (a^2 - r^2)^{1/2} - a \ln\{[a + (a^2 - r^2)^{1/2}]/r\}\right].$$

It is easily verified that $\kappa_\varphi \geq 0$ and $\kappa_r \geq 0$ within the plate, i.e. the solution really corresponds to the corner A.

For the side AB we have $\kappa_r = 0$, i.e. $d^2w/dr^2 = 0$ and $w = A(r - a)$, hence $\kappa_\varphi = -1/r\, dw/dr = A/r$. The constancy of $\bar{D}$ yields then

$$M_\varphi\kappa_\varphi/h_c = \text{const.} \quad \text{or} \quad h_c = K_1 r, \qquad \text{(Fig. 3b)}, \tag{17}$$

(a) (b)

FIG. 3

where K_1 is an arbitrary constant. If the side BC is considered, then $\kappa_r = -\kappa_\varphi$, $w = A \ln a/r$, and the condition $\bar{D}$ = const. leads to the solution $h_c = K_2 r^2$. Since for the states represented by CD a plate cannot deform, there is no solution for this region. For the side AB we have

$$M_\varphi = \sigma_0 K_1^2 r^2 = Cr^2, \qquad M_r = \left(C - \frac{q}{2}\right)\frac{r^2}{3}.$$

Since $M_r = 0$ for $r = a$, $C = q/2$, $M_r = 0$ within the plate, thus the stress state is represented by the point B. The volume of this plate is $V = 0.946\ \pi(q/\sigma_0)^{\frac{1}{2}}\ a^3$ and is greater than that of a plate of constant thickness ($V = 0.815\ \pi(q/\sigma_0)^{\frac{1}{2}}\ a^3$). On the other hand it may be shown that no extremal solution exists for the state BC; thus plates corresponding to BC will have a greater volume than those corresponding to B, which therefore represents a saddle point.

An exact proof of the existence of a minimum for corners and a maximum for sides of the hexagon has been given in Ref. [7] in a somewhat different way.

Example 2. Consider a circular plate symmetrically loaded and made of a material that obeys Mises' yield condition. We then have

$$M_r^2 - M_r M_\varphi + M_\varphi^2 = M_0^2, \qquad (M_r + \delta M_r')^2 - (M_r + \delta M_r')(M_\varphi + \delta M_\varphi') + (M_\varphi + \delta M_\varphi')^2 = M_0^2, \tag{18}$$

$$\kappa_r = \lambda(2M_r - M_\varphi), \qquad \kappa_\varphi = \lambda(2M_\varphi - M_r), \qquad \text{(see [14])}, \tag{19}$$

and hence

$$\delta M_r'\kappa_r + \delta M_\varphi'\kappa_\varphi = \lambda[\delta M_r'(2M_r - M_\varphi) + \delta M_\varphi'(2M_\varphi - M_r)] = -\lambda(\delta M_r'^2 - \delta M_r'\,\delta M_\varphi' + \delta M_\varphi'^2). \tag{20}$$

The quantities $\delta M_r'$ and $\delta M_\varphi'$ can be determined from the yield condition (18) and the equation of equilibrium. Writing

$$\frac{d}{dr}(r\ \delta M_r) - \delta M_\varphi = 0, \qquad \delta M_r = 2M_r \frac{\delta h}{h_c} + \delta M_r', \qquad \delta M_\varphi = 2M_\varphi \frac{\delta h}{h_c} + \delta M_\varphi', \tag{21}$$

we have

$$\frac{d}{dr}(r\ \delta M_r') - \delta M_\varphi' = -2\frac{d}{dr}\left(rM_r \frac{\delta h}{h_c}\right) + 2M_\varphi \frac{\delta h}{h_c}. \tag{22}$$

Equations (18) and (22) determine $\delta M_r'$ and $\delta M_\varphi'$. From these equations it follows that $\delta M_r'$ and $\delta M_\varphi'$ will be of the same order as δh, provided $d(\delta h)/dr$ will be also of the same order. In view of (20) $\delta M_{ij}'\ \kappa_{ij}^c$ is therefore of the second order and an extremum of the volume is obtained.

Since the expression (14) represents the Weierstrass' function for the volume of a plate, we can use $\delta M_r'$ as the variation rather than δh; note that $\delta M_r'$ vanishes at the edge. We can then apply the known criterion for a minimum; setting $M_r' = dM_r/dr$, we have

$$\frac{\partial^2 F}{\partial M_r'^2} = \frac{\partial^2[(M_r^2 - M_r M_\varphi + M_\varphi^2)^{1/2} r]}{\partial M_r'^2} \geq 0, \tag{23}$$

provided the Jacobi condition is satisfied. On performing the calculations, we obtain

$$5M_r^2 + 4M_r M_\varphi - 4M_\varphi^2 \geq 0, \tag{24}$$

and hence

$$-0.725 \leq M_\varphi/M_r \leq 1.725$$

The dashed region shown in Fig. 4 represents the inequalities (25). If the solution

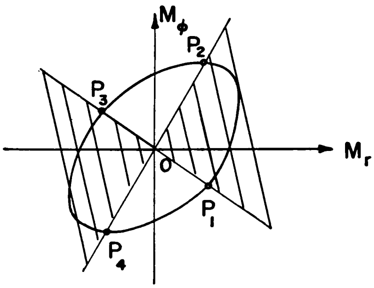

Fig. 4

lies within this region, it will correspond to the minimum of the volume. A problem of optimal design using Mises' yield condition has been studied by Freiberger and Tekinalp [3]. It can be easily checked, that the solution obtained by them is contained within the angle P_1OP_2 (Fig. 4). It is possible, however, that in the region P_2OP_3 there is another solution, for which the volume attains its maximum or corresponds to a saddle point.

If an extremal solution exists, which lies in the region P_1OP_2 then there is no other solution in this region. Indeed, in the transition from one minimal solution to the other, we must pass the maximum, i.e. come into the region P_2OP_3 and then move back. This implies a contradiction, since we would have passed the same minimal state twice. The other minimal solution, if it exists, must therefore lie in the region OP_3P_4 . Furthermore, since there is no upper bound for the volume, if a maximum exists at all, some other minimal solution must also exist, unless we have a saddle point.

3. Design of shells. Assuming that the thickness of a shell is to be altered symmetrically with respect to the median surface, we cannot obtain $\bar{D}$ = const. on both surfaces. We should therefore apply the condition (7). Performing calculations similar to those presented for plates, we obtain the following results. When

$$\Delta = \sigma_{ij}^{+}(\epsilon_{ij}^{c} + \kappa_{ij}^{c} h_c) + \sigma_{ij}^{-}(\epsilon_{ij}^{c} - \kappa_{ij}^{c} h_c) = \text{const.} \tag{26}$$

at every point of a shell, we have (neglecting orders higher than second)

$$\Delta\, \delta V = -\frac{1}{2}\int_A \kappa_{ij}^{c}(\sigma_{ij}^{+} - \sigma_{ij}^{-})\, \delta h^2\, dA - \int_A \left[\int_{-h_c}^{+h_c} \delta\sigma_{ij}(\epsilon_{ij}^{c} + \kappa_{ij}^{c} z)\, dz\right] dA, \tag{27}$$

where σ_{ij}^{+}, σ_{ij}^{-} denote the stresses at the two surfaces and ϵ_{ij}^{c}, κ_{ij}^{c} are the strain rates and rates of curvature of the median surface. The first term in (27) is always positive, and the second, negative. Similarly as before, if $\delta\sigma_{ij} \sim \delta h$, we conclude that condition (26) assures an extremum of the volume. Examples may also be provided which show that in some cases the extremal solutions are maxima.

Problems of optimal design for shells have been studied in Refs. [10], [11], [12], [13]. In Ref. [12], the criterion (26) has been established as sufficient for a minimum of the volume, which is generally true for sandwich shells.

If we assume that the thickness of the shell may be changed on one surface only, while the other surface is kept unaltered, we must take into consideration that the median surface is changed; the problem therefore requires separate treatment. If, for instance, $\bar{D}$ = const. on the surface that is changed, is smaller than $\bar{D}$ on the other surface, our shell will attain an absolute minimum according to the theorem proved above.

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