In this case the vanishing of the strain rate components $\epsilon_{\alpha\beta}$ implies also that the deviator components $h_{\alpha\beta}$ are stationary. Moreover, a limited inverse is also true; that is, the vanishing of $e_{\alpha\beta}$ implies that the strain rate is isotropic.

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STEADY SPHEROIDAL VORTICES—MORE EXACT SOLUTIONS TO THE NAVIER-STOKES EQUATION*

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Abstract. The vorticity equation, the curl of the Navier-Stokes equation, is considered in ellipsoidal coordinates. The steady spheroidal vortex solutions are demonstrated as examples of a class of exact flow solutions characterized by a simple linear vorticity distribution.

Introduction. The Navier-Stokes equation expresses the conservation of momentum in a form suitable for continuum fluid mechanics. For incompressible fluid flow the system of the continuity equation plus the Navier-Stokes equation has a solution that is completely determined, in principle. However, solutions that satisfy the equations exactly are rare due to the non-linear convective term. One such solution that should be included in the meager list of exact solutions is the Hill-Hadamard spherical vortex.

The spherical vortex is an example of a class of steady axi-symmetric flow solutions consisting of a rotational part where the vorticity is proportional to the distance from the axis and an irrotational part. Other examples of this class will be given, in particular, the steady oblate spheroidal vortices which appear to be physically significant.

Hill-Hadamard spherical vortex. Hill [1] showed in 1894 that the stream function

$$\psi = \sin^2 \theta(Ar^2 + Br^4) \tag{1}$$

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satisfied the inviscid condition for steady motion (Lamb [2], p. 245)

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi} = \mathsf{D}\psi = \varpi^2 f(\psi), \tag{2}$$

where (x, ϖ, ϕ) is the cylindrical polar coordinate system. The relation to the spherical polar coordinate system (r, θ, ϕ) is given by $x = r\cos\theta, \varpi = r\sin\theta$. Hill's solution takes $f(\psi)$ as a constant $= -10B$.

Hadamard [3] (and Rybczynsky [4]) in 1911, while discussing the viscous motion of a fluid sphere within another fluid, used the same stream function (1). Stokes [5] had given

$$\psi = \sin^2\theta(Ar^2 + Br^4 + Cr + Dr^{-1}) \tag{3}$$

as a solution to the steady viscous vorticity equation

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial\varpi^2} - \frac{1}{\varpi}\frac{\partial}{\partial\varpi}\right\}^2\psi = \mathsf{D}^2\psi = 0. \tag{4}$$

The physical requirement of finite velocities as $r \to 0$ eliminated the two lower powers of r.

Thus it has been known for a long time that this stream function (1) independently satisfies a condition derived from the Navier-Stokes equation under the assumption of purely inviscid flow (2) and the condition derived for purely viscous flow (4). It follows, though never previously shown to the author's knowledge, that the same stream function satisfies the complete Navier-Stokes equation. Therefore, it is an exact solution.

The velocity field follows from the derivatives of ψ. $\mathbf{q} = u\mathbf{i}_x + v\mathbf{i}_\varpi$ where

$$u = -\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}, \qquad v = \frac{1}{\varpi}\frac{\partial\psi}{\partial x}.$$

The pressure field is found by integration of the momentum equation written in the "vorticity form"

$$\frac{\partial\mathbf{q}}{\partial t} - \mathbf{q}\times\boldsymbol{\omega} = -\nabla\left(\frac{p}{\rho} + \Omega + \frac{1}{2}\mathbf{q}\cdot\mathbf{q}\right) - \nu\nabla\times\boldsymbol{\omega} \tag{5}$$

with the vorticity vector $\boldsymbol{\omega} = \nabla\times\mathbf{q} = \omega\mathbf{i}_\phi = (1/\varpi\mathsf{D}\psi)\mathbf{i}_\phi$. Whence

$$\begin{aligned}\frac{p}{\rho} + \Omega + \frac{1}{2}q^2 &= 10B\{A\varpi^2 + B\varpi^2(x^2 + \varpi^2) - 2\nu x\}\\ &= 10B\{(Ar^2 + Br^4)\sin^2\theta - 2\nu r\cos\theta\}.\end{aligned}$$

For a spherical vortex $\psi = 0$ at $r = a$. This requires $A = -Ba^2$. The Hill-Hadamard spherical vortex is one of the class of flow solutions consisting of a rotational part where the vorticity ω is proportional to ϖ and an irrotational part. Spherical vortices have been demonstrated in real fluids by Spells [6] and Savic [7].

Poiseuille flow. A more familiar example of this class of exact flow solutions with a linear vorticity distribution is the well-known Poiseuille flow. This is often quoted as an exact solution, but the vorticity distribution receives little attention. Its stream function can be written (a = pipe radius)

$$\psi = A\varpi^2(a^2 - \varpi^2). \tag{6}$$

This can be rewritten into a rotational part and an irrotational part as

$$\psi = \left\{-\frac{6A}{5}r^4\sin^2\theta\right\} + \left\{Aa^2r^2C_2^{-1/2}(\cos\theta) + \frac{A}{5}r^4C_4^{-1/2}(\cos\theta)\right\}.$$

(vorticity $\omega = 12\,Ar\sin\theta$)

The irrotational part is given in terms of the Gegenbauer polynomials $C_n^{-1/2}(\cos\theta)$ after Savic [7] rather than the more usual Legendre polynomial form for reasons that will become apparent in the next section. The validity of this solution for real fluids in laminar flow is well-established.

Exact solution in oblate ellipsoidal coordinates. It will be demonstrated that the steady spheroidal vortex is another example of the same class of exact flow solutions. For this purpose the pertinent equations are first transformed to the oblate ellipsoidal coordinate system (η, β, ϕ) by means of the relations

$$\begin{aligned} x &= \sinh\eta\cos\beta \\ y &= \cosh\eta\sin\beta\cos\phi \\ z &= \cosh\eta\sin\beta\sin\phi \qquad (\varpi = \cosh\eta\sin\beta) \end{aligned} \tag{7}$$

The continuity equation becomes (see Goldstein [8], p. 114)

$$\frac{\partial}{\partial\eta}[\cosh\eta\sin\beta(\sinh^2\eta+\cos^2\beta)^{1/2}u_\eta] + \frac{\partial}{\partial\beta}[\cosh\eta\sin\beta(\sinh^2\eta+\cos^2\beta)^{1/2}v_\beta] = 0 \tag{8}$$

for $\mathbf{q} = u_\eta\mathbf{i}_\eta + v_\beta\mathbf{i}_\beta$.

Define a current stream function ψ by

$$\begin{aligned} u_\eta &= -\frac{(\cosh\eta\sin\beta)^{-1}}{(\sinh^2\eta+\cos^2\beta)^{1/2}}\frac{\partial\psi}{\partial\beta}, \\ v_\beta &= \frac{(\cosh\eta\sin\beta)^{-1}}{(\sinh^2\eta+\cos^2\beta)^{1/2}}\frac{\partial\psi}{\partial\eta}, \end{aligned} \tag{9}$$

to identically satisfy (8). The steady vorticity equation can then be written (see [8], p. 115)

$$-\frac{(\cosh\eta\sin\beta)^{-1}}{(\sinh^2\eta+\cos^2\beta)}\left\{\frac{\partial(\psi, \mathsf{D}\psi)}{\partial(\eta,\beta)} - \frac{2\,\mathsf{D}\psi}{\cosh\eta\sin\beta}\frac{\partial(\psi, \cosh\eta\sin\beta)}{\partial(\eta,\beta)}\right\} = \nu\,\mathsf{D}^2\psi, \tag{10}$$

where

$$\mathsf{D} = (\sinh^2\eta+\cos^2\beta)^{-1}\left(\frac{\partial^2}{\partial\eta^2} - \tanh\eta\frac{\partial}{\partial\eta} + \frac{\partial^2}{\partial\beta^2} - \cot\beta\frac{\partial}{\partial\beta}\right).$$

Let

$$\psi = \psi_1\,(\text{irrotational}) + \psi_2\,(\text{rotational}). \tag{11}$$

The irrotational part ψ_1, where

$$\omega = \frac{1}{\varpi}\mathsf{D}\psi_1 = 0$$

represents a trivial solution to (10) but it is convenient to discuss this first.

Assume $\psi_1 = f(\eta)g(\beta)$ and $\mathsf{D}\psi_1 = 0$ is separable for $n(n-1) > 0$ into the pair of ordinary differential equations

$$f''(\eta) - \tanh\eta f'(\eta) - n(n-1)f(\eta) = 0; \qquad g''(\beta) - \cot\beta g'(\beta) + n(n-1)g(\beta) = 0.$$

By applying the transformation $i\sinh\eta = z$ to the former and $\cos\beta = z$ to the latter, *both* equations reduce to a Gegenbauer differential equation (see Magnus and Oberhettinger [9], p. 76)

$$(1-z^2)P''(z) + n(n-1)P(z) = 0. \tag{12}$$

A solution for this is $P = C_n^{-1/2}(z)$, the Gegenbauer polynomial of order n and degree $(-\frac{1}{2})$.

Thus a general solution ψ_1 valid for regions containing the origin is

$$\psi_1 = \sum_{n=2}^{\infty} A_n C_n^{-1/2} (\cos \beta) C_n^{-1/2} (i \sinh \eta). \tag{13}$$

The $n = 0, 1$ terms have been dropped because they lead to velocity singularities along the axis. The Gegenbauer functions of the second kind which also satisfy Eq. (12) lead to infinite velocities as $\beta \to \pi/2$ or $\eta \to 0$ and thus do not appear as part of the physical flow solution. However, the Gegenbauer functions of the second kind with arguments a function of η would be important for other flow problems than the type considered here.

Consider now a rotational part

$$\psi_2 = B \sin^2 \beta \cosh^2 \eta (\cos^2 \beta - \sinh^2 \eta). \tag{14}$$

This makes both the inviscid part of the vorticity equation and the viscous part identically zero. $\omega\varpi = \mathsf{D}\psi_2$ can be transformed by $\cos \beta = \mu$, $i \sinh \eta = z$ into

$$(\mu^2 - z^2)^{-1}\left[(1 - \mu^2) \frac{\partial^2}{\partial \mu^2} - (1 - z^2) \frac{\partial^2}{\partial z^2}\right] B(1 - \mu^2)(1 - z^2)(\mu^2 + z^2).$$

Carrying out the indicated operation, this equals

$$\begin{aligned} -10B(1 - \mu^2)(1 - z^2) &= -10B \sin^2 \beta \cosh^2 \eta \\ &= \kappa\varpi^2. \end{aligned}$$

(Or $\omega = \kappa\varpi$.)

The inviscid part of the vorticity equation (10) gives

$$-\frac{(\cosh \eta \sin \beta)^{-1}}{(\sinh^2 \eta + \cos^2 \beta)} \left\{ \kappa \begin{vmatrix} \dfrac{\partial \psi_2}{\partial \eta} & \dfrac{\partial \psi_2}{\partial \beta} \\[2ex] \dfrac{\partial \varpi^2}{\partial \eta} & \dfrac{\partial \varpi^2}{\partial \beta} \end{vmatrix} - 2\kappa\varpi \begin{vmatrix} \dfrac{\partial \psi_2}{\partial \eta} & \dfrac{\partial \psi_2}{\partial \beta} \\[2ex] \dfrac{\partial \varpi}{\partial \eta} & \dfrac{\partial \varpi}{\partial \beta} \end{vmatrix} \right\}$$

which is identically zero. The viscous part $\nu\mathsf{D}^2\psi_2$ is also satisfied because the term equivalent to $\mathsf{D}\psi_2$ appears in the general solution ψ_1 for $n = 2$ or $\mathsf{D}^2\psi_2 = 0$.

The oblate spheroidal vortex. The exact solution $\psi = \psi_1 + \psi_2$ with the boundary conditions $\psi = 0$ on the ellipsoid $\eta = \eta_0$ and that there be stagnation points on the central axis yields:

$$\psi = C \sin^2 \beta \cosh^2 \eta \left\{ \sinh^2 \eta_0 - \sinh^2 \eta + \frac{\cos^2 \beta}{\sinh^2 \eta_0} (\sinh^2 \eta_0 - \sinh^2 \eta) \right\}. \tag{15}$$

This is a spheroidal vortex. The velocity follows directly from the derivatives of ψ and the pressure field can be obtained by integration of the momentum equation in the vorticity form. The component momentum equations in this coordinate system are

$$\begin{aligned} -v_\beta \omega = &-(\sinh^2 \eta + \cos^2 \beta)^{-1/2} \frac{\partial}{\partial \eta} \left(\frac{p}{\rho} + \Omega + \tfrac{1}{2}[u_\eta^2 + v_\beta^2] \right) \\ &+ \nu (\sinh^2 \eta + \cos^2 \beta)^{-1/2} \left(\omega \cot \beta + \frac{\partial \omega}{\partial \beta} \right), \\ u_\eta \omega = &-(\sinh^2 \eta + \cos^2 \beta)^{-1/2} \frac{\partial}{\partial \beta} \left(\frac{p}{\rho} + \Omega + \tfrac{1}{2}[u_\eta^2 + v_\beta^2] \right) \\ &-\nu (\sinh^2 \eta + \cos^2 \beta)^{-1/2} \left(\omega \tanh \eta + \frac{\partial \omega}{\partial \eta} \right). \end{aligned}$$

The pressure field is

$$\frac{p}{\rho} + \Omega + \tfrac{1}{2}(u_\eta^2 + v_\beta^2) = \kappa\{-\psi + 2\nu \cos\beta \sinh \eta\},$$

where $\kappa = -10C$ is the proportionality factor between the vorticity ω and the axial distance ϖ.

Transforming back to the cylindrical coordinate system

$$\psi = C\varpi^2 (\cosh^2 \eta_0 - \varpi^2 - x^2 \coth^2 \eta_0), \tag{16}$$

where $\psi = 0$ on the oblate ellipsoid

$$x^2 (\sinh^2 \eta_0)^{-1} + \varpi^2 (\cosh^2 \eta_0)^{-1} = 1.$$

An example is shown in Fig. 1 for an ellipsoid with fineness ratio $= \tanh \eta_0 = .5 (\eta_0 = 0.55)$. The velocity components parallel and normal to the axis are

$$u = -2C (\cosh^2 \eta_0 - 2\varpi^2 - x^2 \coth^2 \eta_0),$$

$$v = -2C(x\varpi \coth^2 \eta_0).$$

In the axial plane, $x = 0$, note a circle of zero velocity. This center of circulation occurs at the same fixed ratio of the semi-major radius, $(2)^{-1/2}$, for all values of η_0 .

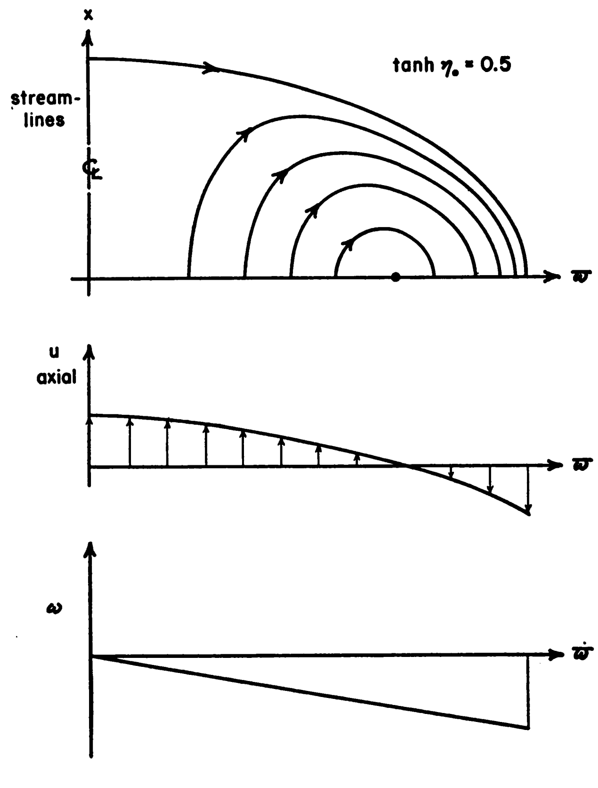

Fig. 1
An example of a spheroidal vortex

Real vortices often have boundaries that can be approximated by oblate spheroids. A comparison of some real flows with theory is given in another place, O'Brien [10]. The measured centers of circulation of quasi-steady vortices over a fairly wide Reynolds number range compare reasonably well with the theoretical value given here.

A more general viscous solution. Neglecting the inviscid part of the steady vorticity equation, a rather general solution, ψ_2 , can be determined from the relation $\mathsf{D}\psi_2 = \psi_1$. Although the equation is not separable, the rotational solution in oblate ellipsoidal coordinates (valid for the region containing $\eta = 0$) has been found to be

$$\psi_2 = \sum_{n=2}^{\infty} B_n C_n^{-1/2} (\cos \beta) C_n^{-1/2}(i \sinh \eta)(\cos^2 \beta - \sinh^2 \eta). \tag{17}$$

But for $n \geq 3$ this stream function does not simultaneously satisfy the inviscid part.

The prolate spheroidal vortex. In similar fashion the problem can be reformulated in prolate ellipsoidal coordinates. The stream function for a prolate spheroidal vortex is

$$\psi = C\varpi^2 (\sinh^2 \xi_0 - \varpi^2 - x^2 \tanh^2 \xi_0). \tag{18}$$

In analogy to the oblate cases, the center of circulation is at the same fixed ratio, $(2)^{-1/2}$, of the midsection radius for all ξ_0 .

Conclusion. It is not known whether there are other members of this class of flow solutions. The second order Laplacian equation that governs "perfect" fluid flow is separable in just eleven coordinate systems. The fourth order purely viscous vorticity equation (4) is not even separable in the two coordinate systems considered here (oblate ellipsoidal and prolate ellipsoidal). Assume, by use of another coordinate system, a new solution to this linear equation is found. If it has vorticity proportional to the axial distance, ϖ, it will be another exact solution.

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