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THREE-DIMENSIONAL FLOWS INSIDE A CYLINDER*

BY

THOMAS C. BENTON

The Pennsylvania State University

Introduction. It is the purpose of this work to derive functions which represent the flow of a perfect fluid inside a cylindrical wall. The velocity perpendicular to this wall must be zero everywhere. Some of the problems discussed will have the axis of the cylinder as an axis of symmetry, but several problems are discussed where the flow is not symmetric about an axis.

In Sec. 1 a certain set of boundary conditions is shown to lead to a series involving Bessel functions. This solution is shown in Sec. 2 to be the result of a point sink at the origin and a circular ring source lying in a plane perpendicular to the axis of the enclosing cylinder and through the origin. This discussion enables the stream function for a source alone and a ring alone to be found. The problem of the combined ring source and point sink in a uniform stream through the cylinder is next discussed. The stream function for a point source in the cylinder not on its axis is then found in Sec. 3.2.

In Sec. 3.4 a new result is obtained which shows that the potential due to a sink not on the axis can be decomposed into a sum of terms, each term being the potential due to a ring source circular in shape but with non-constant strength. It would seem that in many ways these particular ring sources offer the simplest possible configurations with which to begin the study of fields which do not possess axial symmetry.

In Sec. 4 the field of a source and sink on the axis in a stream inside the cylinder is discussed and the semi-axes of the resulting closed stream-surface are computed.

In Sec. 5 the field of the ring with strength $M \cos \theta$ is worked out in detail, this being the least complicated of the ring sources of Sec. 3.4.

1. Let $\psi(\rho, z)$ be a stream function in cylindrical coordinates for a field which is symmetric about the z-axis. Then $\psi(\rho, z)$ is a solution of the equation

$$\frac{\partial^2 \psi}{\partial \rho^2} - \frac{1}{\rho}\frac{\partial \psi}{\partial \rho} + \frac{\partial^2 \psi}{\partial z^2} = 0$$

which has solutions of the type

$$\psi(\rho, z) = \sum_{m=1}^{\infty} A_m \exp(-j_{1m} z/a) \rho J_1(j_{1m}\rho/a),$$

where j_{1m} is the mth positive zero of the Bessel function of the first order $J_1(x)$. It follows

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that $\psi(a, z) \equiv 0$ for all z since $J_1(j_{1m}) = 0$, and $\psi(0, z) \equiv 0$ for all z since ρ is a factor in each term. There is an expansion theorem that if $f(t)$ be a function arbitrarily defined in the interval (0, 1) and

$$\int_0^1 t^{1/2} f(t)\, dt$$

exists, and if

$$a_m = \{2/[J_2(j_{1m})]^2\} \int_0^1 t f(t) J_1(j_{1m} t)\, dt;$$

then

$$\sum_{m=1}^{\infty} a_m J_1(j_{1m} x)$$

converges and its sum is (1/2) $\{f(x + 0) + f(x - 0)\}$ at any internal point x of an interval (a, b) such that $0 < a < b < 1$ and $f(t)$ has limited total fluctuation in (a, b).*

Consider the function $F(\rho)$

$$F(\rho) = \begin{cases} M, & 0 < \rho < ka, \qquad k < 1 \\ 0, & ka < \rho < a, \end{cases}$$

then $F(\rho)/\rho$ has limited total fluctuation in any interval $0 < \epsilon < b \leq 1$, and

$$\int_0^1 t^{1/2}[F(t)/t]\, dt = \int_0^k M t^{-1/2}\, dt = 2Mt^{1/2} \Big|_0^k = 2Mk^{1/2}$$

exists (where $t = \rho/a$).

Therefore, the theorem applies to this function and

$$\begin{aligned} a_m &= 2/[aJ_2^2(j_{1m})] \int_0^1 F(at) J_1(j_{1m} t)\, dt, \\ &= 2M/[aJ_2^2(j_{1m})] \int_0^k J_1(j_{1m} t)\, dt, \\ &= 2M[1 - J_0(j_{1m} k)]/a j_{1m} J_2^2(j_{1m}), \end{aligned}$$

since

$$\int J_1(u)\, du = -J_0(u).$$

Now with $A_m = a_m$ the solution for ψ becomes

$$\psi(\rho, z) = \rho \sum_{m=1}^{\infty} \{2M[1 - J_0(j_{1m} k)]/[a j_{1m} J_2^2(j_{1m})]\} J_1(j_{1m}\rho/a) \exp\,[(-j_{1m} \mid z \mid)/a] \tag{1.1}$$

and

$$\begin{aligned} \psi(\rho, 0) &= \rho \sum_{m=1}^{\infty} \frac{2M[1 - J_0(j_{1m} k)]}{a j_{1m} J_2^2(j_{1m})} J_1(j_{1m}\rho/a) \\ &= \rho F(\rho)/\rho = F(\rho) = \begin{cases} M, & 0 < \rho < k\,a. \\ 0, & ka < \rho < 1 \end{cases} \end{aligned}$$

*G. N. Watson, *Bessel functions*, Cambridge Univ. Press, 1922, Sec. 18.24.

It should be noted that $J_2^2(j_{1m}) = J_0^2(j_{1m})$ so that only values of J_0 and J_1 are needed. Hence we have a flow such that $\psi = 0$ consists of the cylinder $\rho = a$ and the part of the plane $z = 0$ between $\rho = ka$ and $\rho = a$, and the z axis. $\psi = M$ consists of the part of $z = 0$ inside $\rho = ka$. The plot of the values of the function shows that the origin is a sink into which the fluid flows and there is a ring source of radius ka in the plane $z = 0$. (Fig. 1* was made for $k = 1/3$, $M = \pi$.)

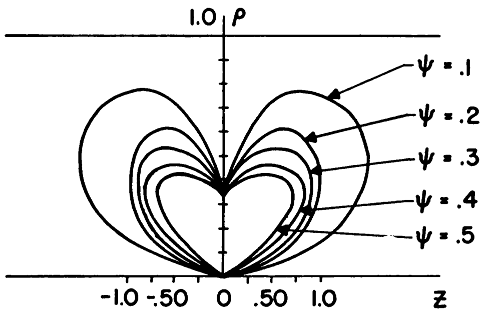

FIG. 1

2. If we take

$$\psi = -\frac{U}{2}\rho^2 + \psi(\rho, z),$$

where $\psi(\rho, z)$ is given by Eq. (1.1), we will have a stream flowing around the ring source and simple sink at its center. If q_z, q_ρ are the velocity components, we have

$$q_z = -\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} = \sum_{m=1}^{\infty} A_m \exp(-j_{1m}z/a) J_0\left(j_{1m}\frac{\rho}{a}\right)\frac{j_{1m}}{a} \tag{2.1}$$

$$q_\rho = \frac{1}{\rho}\frac{\partial\psi}{\partial z} = -\sum_{m=1}^{\infty} A_m \exp(-j_{1m}z/a)\frac{j_{1m}}{a} J_1\left(j_{1m}\frac{\rho}{a}\right). \tag{2.2}$$

If $\rho = 0$, $J_0(0) = 1$, so on the z axis

$$q_z(0, z) = U - \sum_{m=1}^{\infty}\frac{A_m j_{1m}}{a}\exp(-j_{1m}z/a).$$

The stagnation point on the axis will be given by the root of the equation

$$\frac{a^2U}{2M} = \sum_{m=1}^{\infty}\frac{1 - J_0(j_{1m}k)}{J_0^2(j_{1m})}\exp(-j_{1m}z/a) \tag{2.3}$$

when solved for u and $z = ua$. It will be noted that if a stagnation point can occur at a place not on the z-axis q_z and q_ρ both are zero. So from (2.2) a value can be assigned for z and the resulting equation for ρ can be solved, giving the coordinates of the stagnation point. Then these values can be inserted in (2.1) and the value of U/M can be calcu-

*Calculations were done by the Ordnance Research Laboratory computers. See Appendix I for values of ψ, and Fig. 1.

lated, giving the velocity of the stream at a great distance from the origin. It was found that $z = 0.20$ gives an equation

$$\begin{aligned} f(\rho) \equiv\ & 1.054J_1\ (3.8817\rho) + 2.620J_1\ (7.1056\rho) \\ & + 2.825J_1\ (10.1725\rho) + 1.948J_1\ (13.3237\rho) \\ & + 0.972J_1\ (16.4706\rho) + 0.454J_1\ (19.6159\rho) \\ & + 0.278J_1\ (22.7601\rho) + 0.226J_1\ (25.9037\rho) \\ & + 0.167J_1\ (29.0468\rho) + 0.105J_1\ (32.1897\rho) \\ & + 0.058J_1\ (35.3323\rho) + \cdots = 0. \end{aligned}$$

This gives the following table of values:

ρ	0	0.3	0.4	0.49	0.5	0.7
$f(\rho)$	0	2.450	0.668	0.043	−0.020	−0.074

therefore, $z = 0.200$, $\rho = 0.497$ is a stagnation point. This is approximately $z = 0.20$, $\rho = 0.50$. The value of $U/2M = 0.649$ from (2.2); or $U = 1.298m$. Equation (2.3) gives a stagnation point at $-$ 0.65 on the z axis. By direct calculation from (2.0) the values of ψ/M can be calculated.*

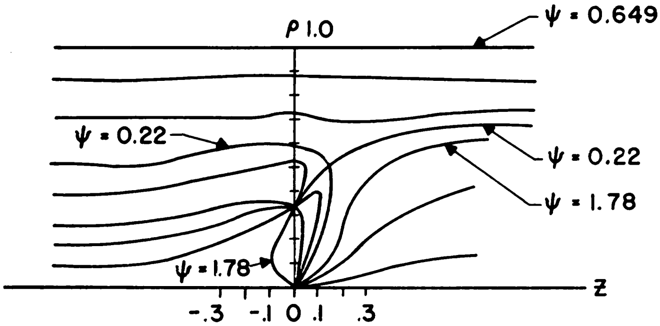

FIG. 2

3. Since the problem of the sink in a cylinder where the flow is along the cylindrical wall is a Neumann problem, it is necessary for $\partial\Phi/\partial n$ to vanish over the cylinder. The velocity potential satisfies the differential equation:

$$\frac{\partial^2\Phi}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\varphi^2} + \frac{\partial^2\Phi}{\partial z^2} = 0.$$

Let

$$\Phi = e^{-kz} \cos s\varphi P(\rho),$$

$$\frac{d^2P}{d\rho^2} + \frac{1}{\rho}\frac{dP}{d\rho} - \frac{s^2}{\rho^2}P + k^2P = 0,$$

$$\rho^2\frac{d^2P}{d\rho^2} + \rho\frac{dP}{d\rho} + (k^2\rho^2 - s^2)P = 0.$$

Therefore, $P = J_s(k\rho)$.

*See Appendix II for a table of values and Fig. 2.

Let

$$\Phi = \sum_k \sum_{s=0}^{\infty} A_{ks} e^{-kz} \cos s\varphi J_s(k\rho)$$

$$\partial\Phi/\partial\rho = \sum_k \sum_{s=0}^{\infty} A_{ks} e^{-kz} (\cos s\varphi) k J_s'(k\rho).$$

Therefore, if $\partial\Phi/\partial\rho = 0$ for $\rho = a$, $J_s'(ka) = 0$. If $h_{s,m}$ is mth root of $J_s' = 0$, $k = h_{s\ m}/a$.

3.1 For any axially symmetric field $s = 0$, $J_0'(ka) = -J_1(ka) = 0$; so

$$ka = j_{1n},$$

$$\Phi = \sum_{n=1}^{\infty} A_n \exp(-j_{1n} \mid z \mid /a) J_0(j_{1n}\rho/a),$$

$$\partial\Phi/\partial z = -\sum_{n=1}^{\infty} A_n (j_{1n}/a) \exp(-j_{1n} \mid z \mid /a) J_0(j_{1n}\rho/a),$$

$$\text{on } z = 0, \quad \frac{\partial\Phi}{\partial z} = \left\{ \begin{matrix} \infty; & \rho = 0, \text{ simple pole} \\ 0; & 0 < \rho \le a \end{matrix} \right\} = -M\,\delta(\rho),$$

$$\int_0^a \int_0^{2\pi} \left(\frac{\partial\Phi}{\partial z}\right)_0 \rho J_0(j_{1r}\rho/a)\, d\theta\, d\rho = -2\pi M \quad \text{if} \quad M$$

is the strength of the sink, because

$$\rho\left(\frac{\partial\Phi}{\partial z}\right)_0 = -M\,\delta(\rho).$$

Hence, the integral $= M J_0(0) \int_0^a\int_0^{2\pi} d\,\theta\, d\rho =$ total flux from the sink on one side of

$$z = 0 = -2\pi A_r \int_0^a (j_{1r}\rho/a) J_0(j_{1r}\rho/a) J_0(j_{1r}\rho/a)\, d\rho,$$

since every integral vanishes except for

$$n = r = -\pi A_r j_{1r} a J_0^2(j_{1r})^*,$$

so

$$A_r = 2M/[j_{1r} a J_0^2(j_{1r})];$$

hence,

$$\Phi = \sum_{n=1}^{\infty} \{2M/[j_{1n} a J_0^2(j_{1n})]\} \exp(-j_{1n} \mid z \mid /a) J_0(j_{1n}\rho/a)$$

is the velocity potential of some field with a sink at the origin with normal velocity 0 on $\rho = a$.

The stream function is

$$\psi = \sum_{n=1}^{\infty} \{2M/[j_{1n} a J_0^2(j_{1n})]\} \rho J_1(j_{1n}\rho/a) \exp(-j_{1n} \mid z \mid /a), \tag{3.1}$$

*G. N. Watson, *Bessel functions*, Cambridge Univ. Press, 1922, Sec. 5.11 (11).

which obviously gives $\psi(a, z) = 0$ (since $J_1(j_{1n}) = 0$). Also note that this is a part of Eq. (1.1). However, this equation gives a velocity 0 at $z = \infty$, so it is necessary to add a stream flowing from right to left; i.e., a term $M\rho^2/a^2$, then the resulting field is that of a sink in a cylinder.

If a stream of velocity U from right to left is added (Fig. 3):

$$\psi = (|z|/z)(2M\rho/a) \sum_{n=1}^{\infty} \{j_{1n}J_0^2(j_{1n})\}^{-1} J_1(j_{1n}\rho/a) \exp(-j_{1n}|z|/a) + \frac{M\rho^2}{a^2} + \frac{U}{2}\rho^2.$$

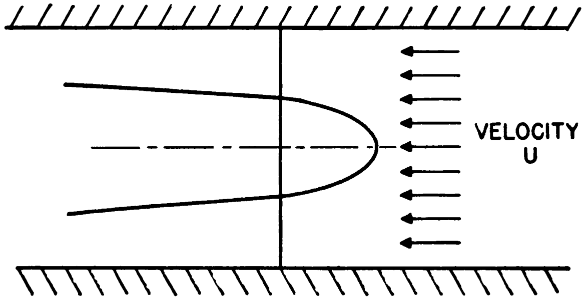

Fig. 3

Assume $a = 1$. Then for $z \geq 0$

$$\psi = 2M\rho \sum_{n=1}^{\infty} \{j_{1n}J_0^2(j_{1n})\}^{-1} J_1(j_{1n}\rho) \exp(-j_{1n}|z|) + \left(M + \frac{U}{2}\right)\rho^2;$$

for $z \leq 0$,

$$\psi = -2M\rho \sum_{n=1}^{\infty} \{j_{1n}J_0^2(j_{1n})\}^{-1} J_1(j_{1n}\rho) \exp(-j_{1n}|z|) + \left(\frac{U}{2} - M\right)\rho^2;$$

$$q_z = -\frac{1}{\rho}\frac{\partial\psi}{\partial\rho} = -\frac{\partial\Phi}{\partial z} = \left\{\sum_{n=1}^{\infty} \{[2MJ_0(j_{1n}\rho)]/J_0^2(j_{1n})\} \exp(-j_{1n}|z|\right\} + 2M - U,$$

$$U > 2M$$

on

$$\rho = 0, \qquad q_z = 0 \quad \text{if} \quad \sum_{n=1}^{\infty} J_0^2(j_{1n}) \exp\left(-j_{1n}|z|\right) = \frac{U}{2M} - 1\cdot$$

Since

$$\frac{J_0^2(j_{1n})}{J_0^2(j_{1,n+1})}\cdot\frac{\exp(-j_{1,n+1}|z|)}{\exp(-j_{1,n}|z|)} \sim \frac{j_{1,n+1}}{j_{1,n}} \exp(-j_{1,n+1} + j_{1,n})|z|$$

$$\cong \left(1 + \frac{\pi}{j_{1n}}\right) \exp(-\pi|z|) \cong \exp(-\pi|z|)$$

as $n \to \infty$, the series converges if $|z| > 0$.

The graph of U/M as a function of z is made from the following Table which was computed from the formula just derived.

z	1	1/2	1/3
U/M	0.150	1.376	3.769

The graph of $U/M = 2 + 2\sum$ can be compared with the function $(U/M)^{1/2}$ which gives the position of the stagnation point for a source in a free stream and it is evident that a stronger stream is needed inside the cylinder to push the stagnation point to the position it has in a free stream. The curves are plotted in Fig. 4.

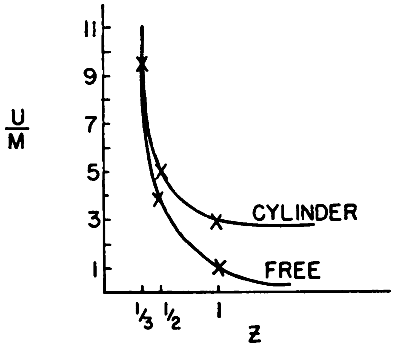

FIG. 4

3.2. In the case where the sink of strength M is at $(b, 0, 0)$ the same method can be used. Let

$$\Phi = \sum_{s=0}^{\infty}\sum_{n=1}^{\infty} A_{ns} \exp(-h_{sn} \mid z \mid /a) J_s(h_{sn}\rho/a) \cos s\rho\varphi, \tag{3.2}$$

where h_{sn} is the nth positive root of $J_s'(z) = 0$. Now

$$(\partial\Phi/\partial z)_{z=0} = -(h_{sn}/a)\Phi(\rho, \varphi, 0).$$

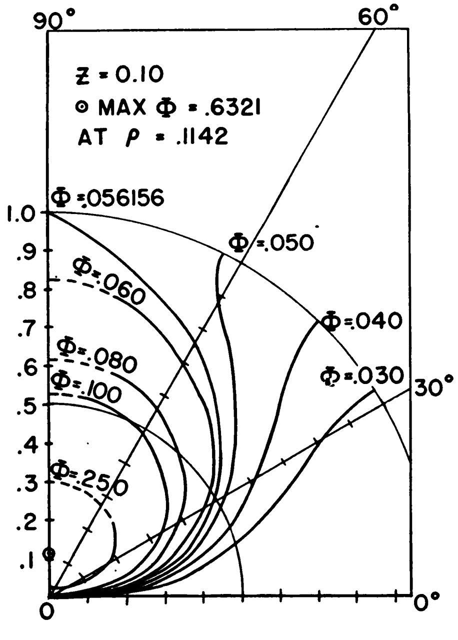

FIG. 5

Hence

$$\int_0^a \int_0^{2\pi} \left(\frac{\partial \Phi}{\partial z}\right)_{z=0} \cos s\varphi J_s(h_{sn}\rho/a)\ \rho\ d\varphi\ d\rho$$
$$= (1 + \delta_{s0})\pi(h_{sn}/a)A_{sn}\cdot(a^2/2)[h_{sn} - \delta^2]h_{sn}^{-2}J_s^2(h_{sn}),$$

where

$$\delta_{s0} = \begin{cases} 1, & s = 0 \\ 0, & s > 0. \end{cases}$$

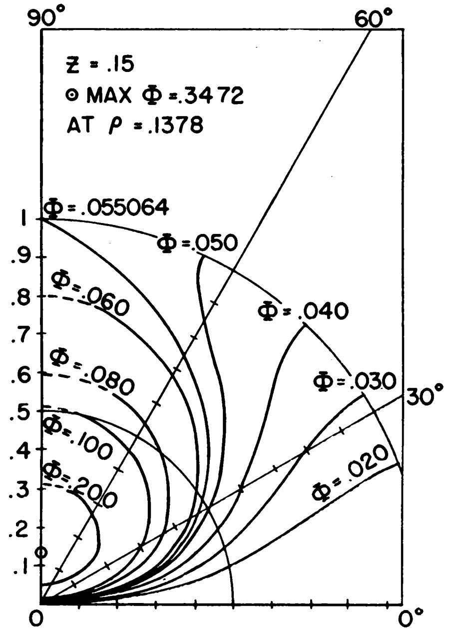

Fig. 6

But $(\partial\Phi/\partial z)_{z=0}$ is zero everywhere except at the point $(b, 0, 0)$, so the integral must be equal to

$$J_s(h_{sn}/a)(-2\pi M).$$

The integral must be interpreted as a Stieltjes Integral of a discontinuous function. Therefore

$$-2\pi M J_s(h_{sn}b/2) = -\pi A_{sn}a(h_{sn}^2 - s^2)(zh_{sn})^{-1}J_s^2(h_{sn})(1 + \delta_{s0}).$$

Therefore

$$A_{sn} = \frac{2Mh_{sn}(2 - \delta_{s0})}{a(h_{sn}^2 - s^2)J_s^2(h_{sn})}\, J_s(h_{sn}b/a).$$

3.3. In the special case where we demand symmetry with respect to the axis we get a set of terms

$$\Phi = 2Ma^{-1}\left\{\sum_{n=1}^{\infty} j_{1n}^{-1}[J_0(j_{1n})]^{-2} \exp\left(-j_{1n} \mid z \mid /a\right) J_0(j_{1n}b/a) J_0(j_{1n}\rho/a)\right\} + 2Mz$$

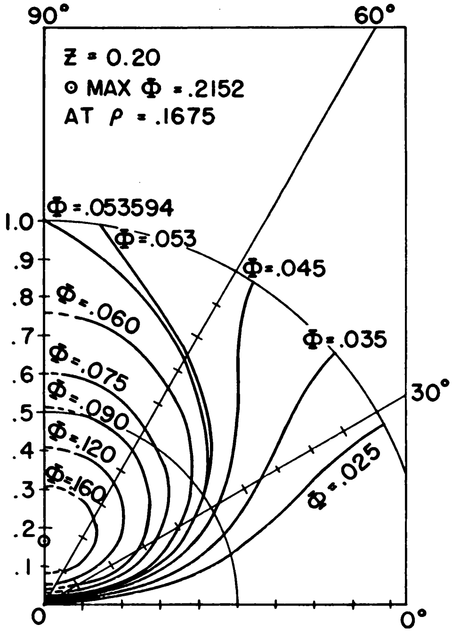

FIG. 7

which still satisfy the differential equation and from (1.1) and (2.1) this must be the potential of a ring sink with $2M$ for its strength and b for its radius, inside a cylinder of radius a after the addition of the stream term. The stream function for this will be

$$\psi = 2Ma^{-1}\left\{\sum_{n=1}^{\infty} j_{1n}^{-1}[J_0(j_{1n})]^{-2} \exp\left(-j_{1n}z/a\right) J_0(j_{1n}b/a) J_1(j_{1n}\rho/a)\right\} + Ma^{-2}\rho^2.$$

3.4. A ring source with variable strength $[M/(2\pi b)] \cos s'\varphi_0$ at the point $(b, \varphi_0, 0)$ gives for the potential of the whole ring

$$\Phi_s = \int_0^{2\pi} M(2\pi)^{-1} \cos s'\varphi_0 \left\{\sum_{s=0}^{\infty}\left[\sum_{n=1}^{\infty} A_{sn} \exp\left(-h_{sn} \,|\, z \,|/a\right) J_s(h_{sn}\rho/a) \cos s(\varphi - \varphi_0)\right]\right\} d\varphi_0 ,$$

but

$$\int_0^{2\pi} \cos s'\varphi_0 \cos s(\varphi - \varphi_0)\, d\varphi_0 = \begin{cases} 0; & s' \neq s, \\ \pi \cos s\varphi; & s' = s; \end{cases}$$

so

$$\Phi_s = (m/2) \cos s\varphi \left\{\sum_{n=1}^{\infty} A_{sn} \exp\left(-h_{sn} \,|\, z \,|/a\right) J_s(h_{sn}\rho/a)\right\}.$$

Since these are exactly the terms of the original series Eq. (3.2.1), which contain cos $s\varphi$, we have the theorem that the potential Φ for a point source at $(b, 0, 0)$ of strength M on a cylinder consists of the sum of the potential Φ_s for $s = 0, 1, 2, \cdots$ where Φ_s is the potential of a ring source of variable strength in the cylinder and passes through the point source and has a strength $[M/(2\pi b)] \cos s\varphi_0$ at the point $(b, \varphi_0, 0)$ of this ring.

The ring for $s = 1$ will be referred to as a ring of the first order. The ring of zero order is just a ring of constant strength since $\cos s\varphi_0 = 1$ if $s = 0$.

In Sec. 5 below, the field of the first order ring will be worked out in detail.

4. Let there be a source of strength M at $\rho = 0$, $z = b$ and a sink of strength M at $\rho = 0$, $z = -b$ and a stream flowing from right to left inside a cylindrical wall with axis along the z axis. If $z > b$ the streaming motions at infinity due to source and sink cancel each other and these terms may be omitted. The same thing occurs for $z < -b$. But between the two singularities $-b < z < b$ there will be a streaming motion from source to sink.

For $z > b$

$$\psi = (U/2)\rho^2 - \sum_{n=1}^{\infty} 2M\rho j_{1n}^{-1}[J_0(j_{1n})]^{-2}J_1(j_{1n}\rho)\{\exp(-j_{1n} \mid z - b \mid) - \exp(-j_{1n} \mid z + b \mid)\}$$

$$= (U/2)\rho^2 - \sum_{n=1}^{\infty} 2M\rho j_{1n}^{-1}[J_0(j_{1n})]^{-2}J_1(j_{1n}\rho)(\sinh j_{1n}b)\exp(-j_{1n}z).$$

If $\psi = 0$, we will have the dividing stream surface so this surface has the equation

$$\rho^2\left\{U/2 - \sum_{n=1}^{\infty} 2M[J_0(j_{1n})]^{-2}\cdot[2J_1(j_{1n}\rho)/(j_{1n}\rho)](\sinh j_{1n}b)\exp(-j_{1n}z)\right\} = 0.$$

If $\rho \neq 0$, the part of this stream surface, not coincident with the axis $\rho = 0$, has for its equation

$$U/(4M) = \sum_{n=1}^{\infty} [J_0(j_{1n}\rho) + J_2(j_{1n}\rho)][J_0(j_{1n}\rho)]^{-2}(\sinh j_{1n}b)\exp(-j_{1n}z)$$

since $(2/z)\,J_1(z) = J_0(z) + J_2(z)$. Now the points on the axis $\rho = 0$ belonging to this surface are the stagnation points, so

$$U/(4M) = \sum_{n=1}^{\infty} [J_0(j_{1n})]^{-2}(\sinh j_{1n}b)\exp(-j_{1n}z)$$

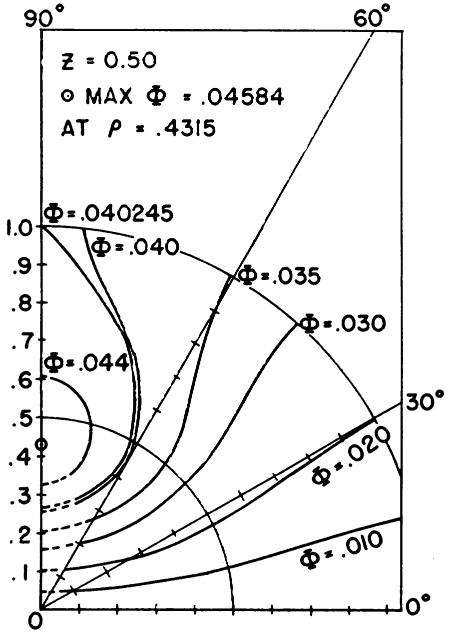

FIG. 8

is an equation for the z coordinate of the stagnation point. This gives the value $U/M = 0.2828$ for $z = 1.50$, and $U/M = 2.6120$ for $z = 1.00$.

For $-b < z < b$,

$$\psi = (U/2)\rho^2 + \sum_{n=1}^{\infty} 2M\rho j_{1n}^{-1}[J_1(j_{1n})]^{-2}J_0(j_{1n}\rho)[\exp(-j_{1n}b)]2\cosh j_{1n}z + 2M\rho^2.$$

Now $\psi = 2M$, $z = 0$ gives the position where the boundary of the closed dividing stream surface crosses the plane $z = 0$. Since $\rho \neq 0$ the equation

$$2M + (U/2) + \sum_{n=1}^{\infty} 2M[J_0(j_{1n})]^{-2}[J_0(j_{1n}\rho) + J_2(j_{1n}\rho)]\exp(-j_{1n}b) = 2M\rho^{-2}$$

gives the intercept. This gives $\rho = 0.955$ for the first case and $\rho = 0.742$ for the second.

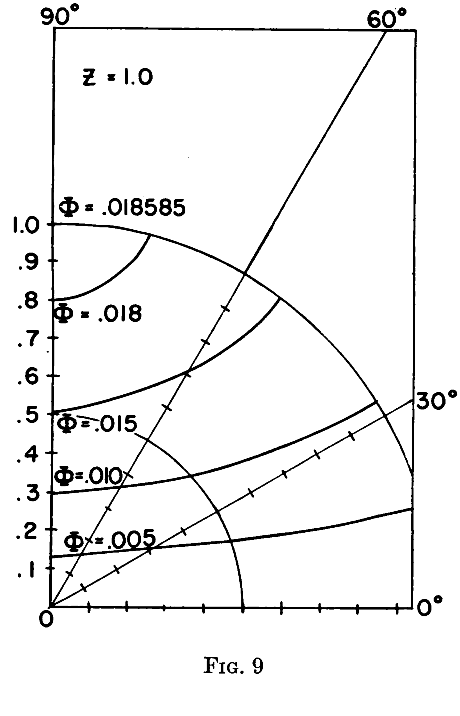

FIG. 9

5. In the case of the most simple ring with variable strength discussed in (3.4), we have

$$\Phi = \tfrac{1}{2}\cos\varphi \sum_{n=1}^{\infty} 2Mh_{1n}J_1(h_{1n}b/a)(h_{1n}^2 - 1)^{-1}[J_1(h_{1n})]^{-2}J_1(j_{1n}\rho/a)\exp(-h_{1n}\,|\,z\,|/a).$$

To obtain some numerical values we will take $b = a/10$, $a = 1$; so that the ring is one tenth of the radius of the cylindrical wall and in a plane perpendicular to the axis of the cylinder with its center on the axis. It is necessary to have the values h_{1n} which are zeros of $J_1'(z)$; since $zJ_1'(z) = zJ_0(z) - J_1(z)$ and the value $z = 0$ is not one in which we

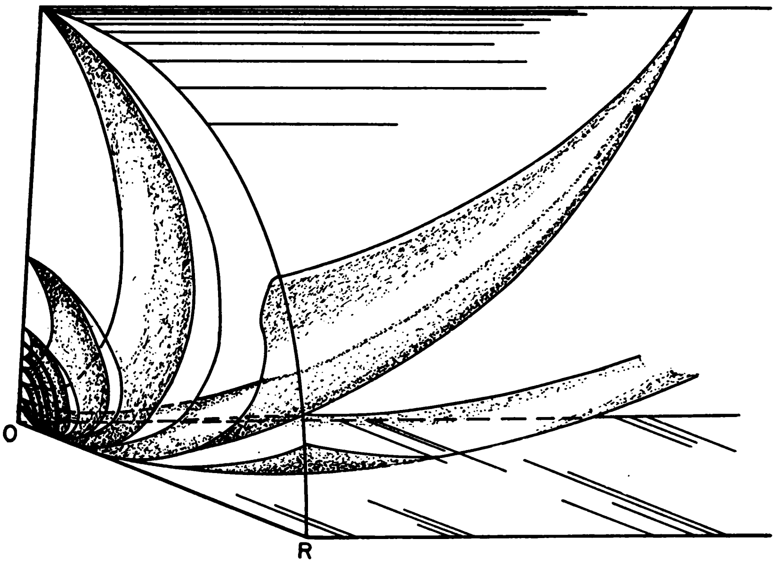

FIG. 10

are interested; these values can be found much more readily from the second form, i.e., $zJ_0(z) = 0$. The values found are as follows

n	1	2	3	4	5	6
h_{1n}	1.8412	5.3314	8.5363	11.7060	14.8636	18.0155
n	7	8	9	10	11	12
h_{1n}	21.1644	24.3113	27.4571	30.6019	33.7462	36.8900

In computing Φ for $z = 0.10$, it was necessary to use 12 terms of the series to get consistent values, a smaller number sufficing for larger values of z.

It will be noted that the strength of the source along the ring varies from a maximum M at $\varphi = 0$, to 0 at $\varphi = \pi/2$, to a minimum $-M$ at $\varphi = \pi$, and back through 0 at $\varphi = 3\pi/2$ to M at $\varphi = 2\pi$. Evidently $\Phi = +\infty$ on the upper half of the ring and $\Phi = -\infty$ on the lower half; and any surface $\Phi = c$ must pass through the points $(1/10, \pi/2, 0)$ and $(1/10, 3\pi/2, 0)$. If c is large and positive, we must obtain a sausage-shaped surface enclosing the upper half of the ring; and if c is large and negative, one enclosing the lower half of the ring. The diametral plane $\varphi = \pi/2, \varphi = 3\pi/2$ is the surface $\Phi = 0$. For small values of Φ there is a closed curve on the cylinder $\rho = a$ which is symmetric to the plane $z = 0$ and to the plane $\Phi = 0$, in which the equipotential surface meets the cylinder orthogonally; and then passes down through the two points of strength 0 on the ring without cutting the plane surface $\Phi = 0$ at any other points. In calculating the values of Φ the values $z = 1.0, 0.5, 0.2, 0.15, 0.1$ were used, and values of ρ from 0 to

1 for about 20 values, so chosen as to give the more interesting situations. Graphs of the equipotentials are given in Figs. 5–9 for each section perpendicular to the axis at the constant values of z in the above series. A sketch of the equipotentials is given in Fig. 10. The streamlines are the curves which are orthogonal to the equipotential surfaces. Since the ring is a curved line-source, the streamlines from any point on the ring must all leave the ring in the plane normal to the ring at that point, and these streamlines form a closed surface which meets the ring again in the point symmetric to the starting point below the surface $\Phi = 0$. These closed surfaces are nested and close down to point limits at the two points of zero strength. They also have as an outer limit, the surface of the bounding cylindrical wall. The streamlines which leave a point on the ring source are shown in Fig. 11.

Acknowledgment. The author wishes to express his thanks to Dr. George F. Wislicenus for directing his attention to the problem and for many helpful discussions as the work progressed. The numerical work would have been almost impossible without the indefatigable aid of the Ordnance Research Laboratory's computing group, particularly Mr. Paul C. Sweetland and Mrs. Barbara R. Kautz. For the figure of the equipotential surfaces of the first order ring, which realizes the form of this solution so completely, the author is deeply indebted to Mrs. Joan P. Lampman.

Appendix I

The stream function

$$\psi(\rho, z) = 2\pi\rho \sum_{m=1}^{\infty} [\exp(-j_n z)][1 - J_0(j_n/3)] j_m^{-1} [J_0(j_m)]^{-2} J_1(j_m \rho)$$

must be evaluated for $0 \leq z$, $0 \leq \rho \leq 1$, where $J_0(z)$ and $J_1(z)$ are the Bessel functions of order zero and one, and j_m is the mth positive zero of $J_1(z)$. Intervals of 0.1 were chosen for ρ and the series was evaluated for $z = .25$ to $z = 2.00$ with intervals of 0.25. This table and that in App. II were computed by the Computing Section of the Ordnance Research Laboratory.

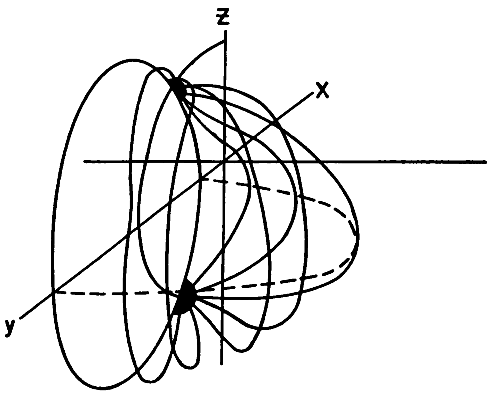

Fig. 11

TABLE 1—ψ

z/ρ	.1	.2	.3	.4	.5	.6	.7	.8	.9
0	1	1	1	0	0	0	0	0	0
.25	.1687	.4502	.5639	.4847	.3437	.2262	.1445	.0867	.0415
.50	.0245	.0814	.1360	.1646	.1637	.1426	.1115	.0763	.0393
.75	.0059	.0212	.0396	.0550	.0631	.0628	.0550	.0408	.0221
1.00	.0018	.0067	.0133	.0195	.0239	.0253	.0255	.0181	.0101
1.25	.0006	.0024	.0047	.0072	.0091	.0099	.0094	.0074	.0042
1.50	.0002	.0009	.0018	.0027	.0035	.0038	.0036	.0029	.0016
1.75	.0001	.0003	.0007	.0010	.0013	.0015	.0014	.0012	.0006
2.00		.0001	.0003	.0004	.0005	.0006	.0005	.0004	.0002

Appendix II

Values of ψ/M Computed from Eq. (2.0)

ρ/z	−.30	−.20	−.15	0−	0+	.10	.15	.20	.30
0	0.000	0.000	0.000	−1.000	1.000	0.000	0.000	0.000	0.000
.1		−0.036		−0.994	1.006	0.147		0.049	
.2		−0.067	−0.052	−0.975	1.025	0.187	0.004	0.114	
.3			−0.102	−0.941	1.059		0.219		
.4	0.171	+0.020	0.000	+0.105	0.105	0.237	0.208	0.188	0.091
.5	0.182	0.104	0.082	0.162	0.162	0.211	0.207	0.220	0.182
.6	0.283	0.198	0.204	0.234	0.234	0.259	0.263		0.283
.7		0.260		0.318	0.318		0.264	0.243	
.8				0.415	0.415				
.9				0.525	0.525				
1.0	0.649	0.649	0.649	0.649	0.649	0.649	0.649	0.649	0.649

A few other values are of interest: For $\rho = 0-$, $Z = 1/3-$, $\psi \sim 1.071$, and for $\rho = 0+$, $Z = 1/3+$, $\psi = 0.071$, ψ has a period 2 and lines for which $\psi = a$ or $a + 2$ etc. are the same line, as illustrated by the line through the stagnation point $\psi = 0.22$ or $\psi = -1.78$.

Appendix III

Figures 5–9 are cross sections of the field of the first order ring at distances $z = 0.1$, 0.15, 0.2, 0.5, 1.0 from the plane of the ring, the curves being the equipotentials. These were made from computations done by the ORL Computing Section.

Figure 10 is a sketch of the general form of the equipotentials in one octant of the cylinder. This is the work of Mrs. Joan Lampman.

Figure 11 is a sketch of the stream lines which emanate from one point of the ring source in the case of the first order ring.