where $h_e^*(z)$ is regular in S_e, including infinity, such that on C

$$\text{re } h_e^*(z) = g(x, y) - k \frac{\partial \phi_{e0}^*}{\partial r} - \frac{l}{a} \frac{\partial \phi_{e0}^*}{\partial \theta} - m\phi_{e0}^*,$$

and $\Omega_e(z)$ is regular in S_i, having singularities only in S_e.

References

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2. N. I. Muskhelishvili, *Some basic problems of the mathematical theory of elasticity*, Translated by J. R. M. Radok, Noordhoff, Groningen, 1953, p. 296
3. E. C. Titchmarsh, *The theory of functions*, 2nd ed., Oxford Univ. Press, 1950, p. 100
4. G. S. S. Ludford, J. Martinek and G. C. K. Yeh, *The sphere theorem in potential theory*, Proc. Camb. Phil. Soc. 51, 389–393 (1955)

ON MATRIX DIFFERENTIAL EQUATIONS*

By JOHN JONES, JR., (*Institute of Technology, Air University*)

The purpose of this note is to obtain a necessary condition and a sufficient condition, of an algebraic nature, for the matrix differential equation

$$AX'' + XB = C, \tag{1}$$

where all matrices considered are n-square, to have a single-valued solution. Capital letters denote matrices and prime denotes the derivative with respect to x. The elements of A, B, C belong to the polynomial domain $\mathfrak{F}[x]$ of the field $\mathfrak{F}$ of real numbers. I is the identity matrix.

Theorem 1. If a solution matrix X of Eq. (1) exists, then the following pair of matrices

$$\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}, \quad \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{2}$$

are equivalent.

Proof. Clearly

$$\begin{bmatrix} I & -X \\ 0 & I \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} I & -X'' \\ 0 & I \end{bmatrix} = \begin{bmatrix} A & C \quad -AX'' \quad -XB \\ 0 & B \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \tag{3}$$

and so the matrices of (2) are equivalent.

Theorem 2. If the matrices of (2) are similar and there exist non-singular constant matrices P, Q such that PAP^{-1}, QBQ^{-1} are diagonal matrices exhibiting the invariant factors a_i, $i = 1, 2, \cdots, \alpha$, and b_j, $j = 1, 2, \cdots, \beta$ of A, B, respectively, along the

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main diagonal, then there exists a solution X of Eq. (1) for $-\infty < a \le x \le b < +\infty$ except at the zeros of the invariant factors of A.

Since the matrices of (2) are similar,

$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} A & C \\ 0 & B \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} = \begin{bmatrix} PAP^{-1} & PCQ^{-1} \\ 0 & QBQ^{-1} \end{bmatrix} = M$$
$$\begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} P^{-1} & 0 \\ 0 & Q^{-1} \end{bmatrix} = \begin{bmatrix} PAP^{-1} & 0 \\ 0 & QBQ^{-1} \end{bmatrix} = N, \tag{4}$$

and the matrices M, N are similar. Form the equation

$$DU'' + UF = G, \tag{5}$$

where $D = PAP^{-1}$, $F = QBQ^{-1}$, $G = PCQ^{-1}$. Consider Eq. (5) element-wise, namely,

$$d_{ii}u''_{ij} + u_{ij}f_{jj} = g_{ij}\,, \qquad (i, j = 1, 2, \cdots, n), \tag{6}$$

where $u''_{ij} \varepsilon U''$, $u_{ij} \varepsilon U$, $d_{ii} \varepsilon D$, $f_{jj} \varepsilon F$, and $g_{ij} \varepsilon G$.

Next to show that Eq. (6) always has a solution. W. E. Roth* has shown that if the matrices of (2) are equivalent, then for the elements g_{ij} of G: (i) g_{ij}, for $1 \le i \le \alpha$, $1 \le j \le \beta$, is a multiple of the greatest common factor of d_{ii} and f_{jj}; (ii) g_{ij} for $1 \le i \le \alpha$, $\beta < j \le n$, is a multiple of d_{ii}; (iii) g_{ij}, for $\alpha < i \le n$, $1 \le j \le \beta$, is a multiple of f_{jj}; (iv) g_{ij}, for $\alpha < i \le n$, $\beta < j \le n$, is identically zero. Thus g_{ij} cannot be different from zero when both d_{ii}, f_{jj} are identically zero, and in each of the four cases above, (6) has a solution.

Let $U = P\,X\,Q^{-1}$, then $U'' = PX''Q^{-1} = (PXQ^{-1})''$, so Eq. (5) may be written as

$$(PAP^{-1})(PX''Q^{-1}) + (PXQ^{-1})(QBQ^{-1}) = PCQ^{-1}. \tag{7}$$

Multiplying (8) on the left by P^{-1} and on the right by Q, we have Eq. (1); thus $X = P^{-1}\,UQ$ is a solution of Eq. (1).

FURTHER PROPERTIES OF CERTAIN CLASSES OF TRANSFER FUNCTIONS: II**

By A. H. ZEMANIAN (*College of Engineering, New York Universiiy*)

This note is a sequel to a previously published paper [1]. The notation and terminology used here is the same as before. The purpose of this note is to point out a consequence of some previously published results [1, 2], which is immediately applicable to rational transfer functions that have no poles in the right-half plane and have at least twice as many poles as zeros. Such transfer functions arise quite commonly in physical systems.

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*W. E. Roth, *The Equation $AX - YB = C$, and $AX - XB = C$ in Matrices,* Proc. Am. Math. Soc. **3**, 392–396 (1952).

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