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QUASI-LINEARIZATION AND UPPER AND LO VER BOUNDS FOR VARIATIONAL PROBLEMS*

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1. Introduction. Since most variational problems are analytically intractable, recourse must be had to approximate techniques if numerical results are desired. One of the most useful devices is that of writing the desired extremum as both a minimum of one quantity and a maximum of another. In this way we obtain both upper and lower bounds and thus estimates of the accuracy of approximations.

In this note we wish to indicate how the technique of quasi-linearization [1], [2] can be used in a number of cases to provide both a minimum and a maximum problem, starting with merely one or the other.

We shall present the technique applied to the simplest type of variational problem. It will be clear from what follows that many other applications can be made.

2. A minimization problem. Consider the problem of minimizing the functional

$$J(u) = \int_0^1 (u'^2 + \varphi(u))\, dt, \tag{1}$$

where u may or may not be subject to end-point conditions such as $u(0) = c$. Let us assume that $\varphi(u)$ is convex for all u so that we can write

$$\varphi(u) = \max_v \, [\varphi(v) + (u - v)\varphi'(v)], \tag{2}$$

see [1], [2].

Then

$$\min_u \int_0^1 (u'^2 + \varphi(u))\, dt = \min_u \max_v \left[\int_0^1 (u'^2 + \varphi(v) + (u - v)\varphi'(v))\, dt \right]. \tag{3}$$

It can now be shown on the basis of quite general results that the minimization and maximization operations in the foregoing equation can be interchanged. Thus,

$$\min_u \int_0^1 (u'^2 + \varphi(u))\, dt = \max_v \min_u \left[\int_0^1 (u'^2 + \varphi(v) + (u - v)\varphi'(v))\, dt \right]. \tag{4}$$

The simplest proof of the interchange of min and max is a direct one, based upon the explicit solution of the minimization problem.

3. Solution of inner problem. The Euler equation of the minimization problem in (2.4) is

$$u'' - \frac{\varphi'(v)}{2} = 0 \tag{1}$$

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with the initial conditions $u(0) = c$, $u'(1) = 0$. Making the change of variable $u = c + w$, we have the functional

$$\int_0^1 (w'^2 + w\varphi'(v) + (c - v)\varphi'(v) + \varphi(v))\, dt \tag{2}$$

and the equation

$$w'' - \frac{\varphi'(v)}{2} = 0, \qquad w(0) = w'(1) = 0. \tag{3}$$

From (3) we obtain

$$\begin{aligned} 0 = \left[\int_0^1 w\left(w'' - \frac{\varphi'(v)}{2}\right) dt = ww'\right]_0^1 - \int_0^1 \left(w'^2 + \frac{\varphi'(v)w}{2}\right) dt \\ = -\int_0^1 \left(w'^2 + \frac{\varphi'(v)w}{2}\right) dt. \end{aligned} \tag{4}$$

Hence, for the minimizing value of the functional in (2), we have the expression

$$\int_0^1 \left(\frac{w\varphi'(v)}{2} + (c - v)\varphi'(v) + \varphi(v)\right) dt, \tag{5}$$

where w is the solution of (3),

$$w = \frac{1}{2}\int_0^1 k(s, t)\varphi'(v(s))\, ds. \tag{6}$$

Here $k(s, t)$ is the Green's function associated with the boundary conditions of (3). Using this expression in (5), we obtain the functional

$$\frac{1}{4}\int_0^1 \int_0^1 k(s, t)\varphi'[v(s)]\varphi'[v(t)]\, ds\, dt + \int_0^1 [(c - v)\varphi'(v) + \varphi(v)]\, dt. \tag{7}$$

4. Equivalent problems. We can thus write

$$\begin{aligned} \min_u \int_0^1 (u'^2 + \varphi(u))\, dt \\ = \max_v \left[\frac{1}{4}\int_0^1 \int_0^1 k(s, t)\varphi'[v(s)]\varphi'[v(t)]\, ds\, dt + \int_0^1 [(c - v)\varphi'(v) + \varphi(v)]\, dt\right]. \end{aligned} \tag{1}$$

The first expression will yield upper bounds; the second will yield lower bounds. As mentioned above, the maximization can be carried through by standard techniques, yielding an "Euler equation" identical with the original.

5. Partial differential equations. In a similar fashion, using a Green's function for the region R, we can convert a minimization problem such as

$$\min_u \int_R (u_x^2 + u_y^2 + e^u)\, dA, \tag{1}$$

where $u = 0$ on the boundary of A, into an equivalent maximization problem.

References

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2. R. Bellman, *Functional equations in the theory of dynamic programming—V: Positivity and quasi-linearity*, Proc. Natl. Acad. Sci. USA **41**, 743–746 (1955)