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DISCONTINUITIES IN INTEGRAL-TRANSFORM SOLUTIONS*

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Summary. Criteria are derived for the determination of the magnitude and the location of discontinuities of solutions in the form of definite integrals obtained by means of integral-transform techniques. The types of integrals arising with the Fourier sine or cosine transforms and those arising with the Laplace transforms are considered in detail. Applications of the theory arise particularly with problems of wave propagation, where interest is centered on the location of wave fronts and the magnitude of jumps there; two illustrative examples of such problems relating to Timoshenko beams are included.

Introduction. In physical problems involving wave propagation phenomena, attention is frequently centered upon the speed of propagation of discontinuities, and on the magnitude of these discontinuities as they progress through the medium. Such information is usually obtained by examining the solution of the particular problem under consideration by *ad hoc* methods, and it would therefore be desirable to have available some general criteria for the detection of the location and magnitude of these discontinuities. The establishment of such criteria is the purpose of this paper, which considers the forms of solutions arising when some types of integral transform techniques are employed. The criteria developed permit the direct determination of the wave-front location and of the local behavior there, from an inspection of the solution in the transform domain, without requiring a detailed inversion.

The developments which follow are divided into four parts: the first contains some preliminary concepts required in the subsequent analyses, the second and third deal with the Fourier sine and cosine and with the Laplace transforms respectively, and the fourth discusses the applications to wave propagation problems.

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1. Preliminary theorems and definitions. For purposes of future reference, it will be convenient to recall first the following two theorems:

I. A sufficient condition[1] for the validity of the equation

$$\int_a^b f(p, x_0 + 0)\, dp = \lim_{x \to x_0} \int_a^b f(p, x)\, dp \tag{1}$$

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[1]See, for example, Hobson [1], vol. II, p. 323. This theorem (as indeed many of the present developments) may be extended to integrals defined in the Lebesgue sense; because of the physical nature of the problems considered here, the discussion in this paper is restricted to Riemann integrals.

provided that $f(p, x)$ be integrable in the interval (a, b), for some interval $x_0 < x \leq x_0 + \alpha$, and that $f(p, x_0 + 0)$ exist, is that a function $\varphi(p) \geq 0$ exist, integrable in (a, b), such that

$$| f(p,x) | \leq \varphi(p) \quad \text{in} \quad a \leq p \leq b, \qquad x_0 \leq x \leq x_0 + \alpha. \tag{1a}$$

Similarly,

$$\int_a^b f(p, \infty) \, dp = \lim_{x \to \infty} \int_a^b f(p, x) \, dp \tag{2}$$

provided that $f(p, x)$ be integrable in (a, b) for $x \geq \alpha$ and that $f(p, \infty)$ exist, if a function $\varphi(p) \geq 0$ exists, integrable in (a, b), such that

$$| f(p,x) | \leq \varphi(p) \quad \text{in} \quad a \leq p \leq b, \qquad x \geq \alpha. \tag{2a}$$

II. A sufficient condition[2] for $f(x)$ to be absolutely integrable in (a, ∞) is that $f(x)$ be integrable in all finite intervals (a, b) and that

$$\lim_{x \to \infty} x^n f(x) = 0 \quad \text{for} \quad n > 1. \tag{3}$$

We will denote the discontinuity of a function $f(x)$ at a point x_0 as [3]$\mathcal{S}(x_0)$, and define it as

$$\mathcal{S}f(x_0) = f(x_0 + 0) - f(x_0 - 0) \tag{4}$$

it being assumed that both $f(x_0 + 0)$ and $f(x_0 - 0)$ exist. Clearly if $\mathcal{S} f(x_0) = 0$ the function $f(x)$, with $f(x_0)$ suitably defined, is continuous at $x = x_0$.

2. Fourier sine and cosine transforms. Given an integral of the form

$$F(x) = \int_0^\infty f(p) \begin{matrix} \sin \\ \cos \end{matrix} g(p, x) \, dp, \tag{5}$$

where it is assumed that (a) $| f(p) |$ is integrable in every finite interval $0 < a \leq p \leq b$, and the (b) a number $p_0 > 0$ can be found such that, for $0 \leq p \leq p_0$,

$$| f(p) \begin{matrix} \sin \\ \cos \end{matrix} g(p, x) | \leq \varphi(p),$$

a positive integrable function; it is now desired to find the value of $\mathcal{S}F(x)$ for all x in $(-\infty, +\infty)$.

We note first, for future convenience, the magnitude of the discontinuities in the following simple special cases of (5). For $a > 0$ and $n > 0$,

$$\mathcal{S} \int_a^\infty \frac{\sin px}{p^n} \, dp \begin{cases} = 0 \quad \text{for} \quad x \neq 0 \\ = \begin{Bmatrix} 0 & \text{if} & n > 1 \\ \pi & \text{if} & n = 1 \\ \infty & \text{if} & n < 1 \end{Bmatrix} \quad \text{for} \quad x = 0 \end{cases} \tag{6}$$

[2]See Hobson [1], vol. I, p. 505, where conditions less stringent than (3) are also given. For instance, if, for $n > 1$, $\lim_{x \to \infty} [x(\log x)^n f(x)] = 0$, the theorem still holds, and it will be readily noticed that Theorems IV and V can be similarly generalized without difficulty, though these generalizations will not be carried out explicitly here.

[3]The symbol $\mathcal{S}$ stands for *saltus*, though the usual definition of saltus is somewhat different from that given here; see [1], vol. I, p. 301. See also footnote 14 of the present paper.

and[4]

$$\mathcal{S}\int_a^\infty \frac{\cos px}{p^n}\,dp = 0 \tag{7}$$

for all x.

III. It will be useful to note that, for any positive number a, the integral

$$\int_0^a f(p) \,{}^{\sin}_{\cos}\, g(p, x)\,dp$$

is a continuous function of x. Write in fact this integral, for any a, as

$$\int_0^{p_1} f(p) \,{}^{\sin}_{\cos}\, g(p, x)\,dp + \int_{p_1}^a f(p) \,{}^{\sin}_{\cos}\, g(p, x)\,dp; \qquad 0 < p_1 \leq p_0$$

where the first integral is continuous by property (b) and I, and the second by property (a) and I.

IV. $F(x)$ defined in Eq. (5) is a continuous function of x, for all x, if

$$\lim_{p\to\infty} p^m f(p) = 0 \quad \text{for} \quad m > 1. \tag{8}$$

Note in fact that, with $a > 0$,

$$F(x) = \int_0^a f(p) \,{}^{\sin}_{\cos}\, g(p, x)\,dp + \int_a^\infty f(p) \,{}^{\sin}_{\cos}\, g(p, x)\,dp, \tag{9}$$

where the first integral has already been seen to represent a continuous function and the second integral is continuous by virtue of Theorem I and since

$$\left| f(p) \,{}^{\sin}_{\cos}\, g(p, x) \right| \leq | f(p) |,$$

the function $f(p)$ being absolutely integrable by Theorem II.

V. If there exist two numbers $n > 0$ and $m > 1$, such that, for some constant K,

$$\lim_{p\to\infty} p^m\left[f(p) - \frac{K}{p^n}\right] = 0 \tag{10}$$

then, with $a > 0$,

$$\mathcal{S}F(x) = K\mathcal{S}\int_a^\infty \frac{1}{p^n} \,{}^{\sin}_{\cos}\, g(p, x)\,dp. \tag{11}$$

This may be seen by writing

$$\begin{aligned} F(x) = \int_0^a f(p) \,{}^{\sin}_{\cos}\, g(p, x)\,dp + \int_a^\infty \left[f(p) - \frac{K}{p^n}\right] {}^{\sin}_{\cos}\, g(p, x)\,dp \\ + K\int_a^\infty \frac{1}{p^n} \,{}^{\sin}_{\cos}\, g(p, x)\,dp \end{aligned} \tag{12}$$

and by then noting that the second of the integrals on the right-hand side, is, by Theorem IV, a continuous function of x. In the special case of $K = 0$, this case of course reduces to that of Eq. (8).

[4]Even though the integral itself in unbounded for $x = 0$, $n < 1$.

VI. The discontinuities in the last integral of Eq. (12) must be studied next, and it will now be shown that

$$\mathcal{S} \int_a^\infty \frac{1}{p^n} \begin{matrix} \sin \\ \cos \end{matrix} g(p, x)\, dp = 0 \quad \text{if} \quad n > 1 \tag{13}$$

and that

$$\mathcal{S} \int_a^\infty \frac{1}{p^n} \begin{matrix} \sin \\ \cos \end{matrix} g(p, x)\, dp = \mathcal{S} \int_a^\infty \frac{1}{p^n} \begin{matrix} \sin \\ \cos \end{matrix} [p\xi(x)]\, dp \quad \text{if} \quad 0 < n \leq 1 \tag{14}$$

provided that a function $\xi(x)$ exists for which

$$\lim_{p \to \infty} p^r[g(p, x) - p\xi(x)] = 0 \quad \text{for some} \quad r > 1 - n. \tag{14a}$$

It may be recalled that the right-hand side of (14) is given expicitly in Eqs. (6) and (7). The validity of (13) is evident from Theorem I, since $(1/p^n)$ is integrable in (a, ∞). For the second case we proceed as follows

$$\int_a^\infty \frac{1}{p^n} [\sin g(p, x) - \sin p\xi]\, dp = 2 \int_a^\infty \frac{1}{p^n} \sin \left[\frac{g(p, x) - p\xi}{2} \right] \cos \left[\frac{g(p, x) + p\xi}{2} \right] dp.$$

A number $\alpha > 0$ can be found, however, such that

$$| g(p, x) - p\xi(x) | < \frac{\epsilon}{p^r} \quad \text{for} \quad p > \alpha.$$

Therefore, for p sufficiently large,

$$\left| \frac{1}{p^n} \sin \left[\frac{g(p, x) - p\xi}{2} \right] \right| \leq \frac{| g(p, x) - p\xi |}{2p^n} < \frac{\epsilon}{2p^{n+r}}$$

and since by hypothesis $n + r > 1$, use of Theorem I completes the proof. Analogous developments hold when the integrand contains the cosine rather than the sine.

VII. The results obtained thus far lead to the following corollaries.

If the functions $f(p)$ and $g(p, x)$ satisfy, in addition to the conditions previously given in conjunction with Eq. (5), also relations (10) and (14a), then[5]

$$\mathcal{S} \int_0^\infty f(p) \sin g(p, x)\, dp \begin{cases} = 0 & \text{for} \quad \xi(x) \neq 0 \\ = \begin{cases} 0 & \text{if} \quad n > 1 \\ K\pi & \text{if} \quad n = 1 \\ \infty & \text{if} \quad n < 1 \end{cases} & \text{for} \quad \xi(x) = 0 \end{cases} \tag{15a}$$

and

$$\mathcal{S} \int_0^\infty f(p) \cos g(p, x)\, dp = 0. \tag{15b}$$

Furthermore, condition (10) is satisfied if the function $f(p)$ admits for large p of a representation of the form

[5] Note that, as in the discussion of footnote 14, the functions on the lefthand side of Eqs. (15) are here regarded as functions of ξ, and that what is really calculated by these equations is the quantity $[F(\xi + 0) - F(\xi - 0)]$.

$$f(p) = \frac{K}{p^n}\left[1 + 0\left(\frac{1}{p}\right)\right]. \tag{16a}$$

Similarly, condition (14a) is satisfied if, for large p,

$$g(p, x) - p\xi(x) = \frac{K_1}{p} + 0\left(\frac{1}{p^2}\right) \tag{16b}$$

for the case in which $0 < n \leq 1$, where K_1 is a constant.

VIII. Calculation of derivatives. Closely allied[6] to the foregoing results is the question of the validity of the relation

$$\frac{dF(x)}{dx} = \int_0^\infty f(p) \frac{\partial g(p, x)}{\partial x} \begin{bmatrix} \cos \\ -\sin \end{bmatrix} g(p, x)\, dp. \tag{17}$$

The validity of (17) rests on the permissibility of the interchange of the integration and of the limiting process involved in the derivative. Since such an interchange is essentially that considered in Theorem I, it is clear that Eq. (17) will hold whenever

$$\mathcal{S} \int_0^\infty f(p) \frac{\partial g(p, x)}{\partial x} \begin{bmatrix} \cos \\ -\sin \end{bmatrix} g(p, x)\, dp = 0. \tag{18}$$

In physical applications it will often occur that Eq. (18) holds everywhere except at a number of isolated points $x_1, x_2, \cdots$. At any one of these points, say x_i, we may calculate $dF/dx\ (x_i + 0)$ and $dF/dx\ (x_i - 0)$ from (17) and hence also $\mathcal{S}\ dF/dx\ (x_i)$. It follows that (17) may be used not only for the calculation of derivatives at any point where (18) holds [and where therefore $\mathcal{S}\ dF/dx = 0$], but also for the calculation of derivative discontinuities at any isolated point where (18) does not hold. This result can be extended to derivatives of orders higher than the first, provided that the integrals in question exist;[7] thus

$$\mathcal{S} \frac{d^nF(x_i)}{dx^n} = \mathcal{S} \int_0^\infty f(p) \frac{\partial^n}{\partial x^n} \left[\begin{matrix} \sin \\ \cos \end{matrix}\, g(p, x)\right] dp. \tag{19}$$

The integral on the right-hand side may expanded by performing the indicated differentiations in the integrand, and it will then appear, in a general case, as the sum of $(n + 1)$ integrals of the form (5); the discontinuities in each of these integrals can be evaluated by the preceding rules.

3. Laplace transforms. When the solution of a particular problem is given in terms of the Laplace transformation, then its discontinuities may be studied either by calculation of the inversion integral along a suitable path,[8] or by a reduction to integrals of the form of that Eq. (5), to which the preceding theory can then be applied. The latter course is followed here.

The Laplace inversion integral is of the form

$$F(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} f(p) e^{g(p,x)}\, dp, \tag{20}$$

[6]See [1], vol. II,. p. 355.

[7]The theorem may be used in cases in which the integrals do not exist, by the procedure described in the last illustrative example at the end of this paper.

[8]The effective tool in this process is Jordan's lemma; see for example reference [2].

where c is a constant chosen so that all the singularities of the integrand occur to the left of the line $p = c + iy$ in the complex plane. The integrand may be decomposed into its real and imaginary parts: on the line

$$p = c + iy \tag{21a}$$

and with notation

$$f(p) = f_R(y) + if_I(y); \qquad g(p, x) = g_R(y, x) + ig_I(y, x) \tag{21b}$$

the result is

$$\begin{aligned} F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} &\exp\,[g_R(y, x)][f_R(y) \cos g_I(y, x) - f_I(y) \sin g_I(y, x)]\, dy \\ &+ \frac{i}{2\pi} \int_{-\infty}^{\infty} \exp\,[g_R(y, x)][f_R(y) \sin g_I(y, x) + f_I(y) \cos g_I(y, x)]\, dy. \end{aligned} \tag{22}$$

It is now clear that the real and imaginary parts each consist of two integrals of the form (5), which can therefore be studied by the techniques used in Part 2 of the present work. Usually however, the imaginary part will be found to vanish (as normally it is expected to do in a physical problem) by reasons of symmetry, while the integrand in the real part is usually, and for the same reasons, an even function of y [3]. Only this case need therefore be considered; for it we then have[9]

$$F(x) = \frac{1}{\pi} \int_{0}^{\infty} \exp\,[g_R(y, x)][f_R(y) \cos g_I(y, x) - f_I(y) \sin g_I(y, x)]\, dy. \tag{23}$$

The following theorem will now be proved.

IX. If equations of the form of (10) and (14a) are satisfied, that is if

$$\lim_{p\to\infty} p^m \left[f(p) - \frac{K}{p^n} \right] = 0, \qquad n > 0, \quad m > 1 \tag{24a}$$

and if [10]

$$\begin{aligned} &\lim_{p\to\infty} p^r [g(p, x) - p\xi(x)] = 0, \qquad r > 1 - n, \quad 0 < n \leq 1 \\ &\lim_{p\to\infty} [g(p, x) - p\xi(x)] = 0, \qquad n > 1 \end{aligned} \tag{24b}$$

[9]It may be of interest to comment here on the usual procedure of obtaining solutions by means of the Laplace transform technique. Usually the integral in (20) is first evaluated by means of a contour integration along a suitable path, and (though in some simple problems an answer in closed form is thus obtained) in most cases this results in several real definite integrals—usually improper—to be evaluated numerically. It is generally felt that the procedure just outlined is preferable to the direct numerical evaluation of (23), because the integrals arising after the contour integration are supposed to be in a form better suited for computation. This may not be the case, however, if the discontinuities of $F(x)$ are first eliminated from the right-hand side of (23), as was done for example in [4]. Furthermore, since the principal drawback of the Laplace transform method is the difficulty of inversion, it would seem that the above alternative procedure should be given serious consideration. For similar reasons, it goes without saying that use of the Fourier sine and cosine transforms, whenever thay are applicable, is preferable to use of the Laplace transform.

[10]Note that conditions (a) and (b) following Eq. (5) hold here because the integrands in question are analytic on the line $p = c + iy$.

then[11]

$$\mathcal{S}F(x)\begin{cases} =0 \quad \text{for} \quad \xi \neq 0 \\ =\begin{cases} 0 & \text{if} \quad n > 1 \\ K & \text{if} \quad n = 1 \\ \infty & \text{if} \quad n < 1 \end{cases} \quad \text{for} \quad \xi = 0, \end{cases} \tag{25}$$

where $F(x)$ is given by Eq. (23).

To prove this, note first that (24a) implies that

$$\begin{aligned} &\lim_{y\to\infty}\left[f_R(y) - \frac{K\cos n\theta}{(c^2+y^2)^{n/2}}\right] = 0 \\ &\lim_{y\to\infty}\left[f_I(y) + \frac{K\sin n\theta}{(c^2+y^2)^{n/2}}\right] = 0 \end{aligned} \tag{26a}$$

and (24b) implies that

$$\begin{aligned} &\lim_{y\to\infty} y^r[g_R(y, x) - c\xi(x)] = 0, \\ &\lim_{y\to\infty} y^r[g_I(y, x) - y\xi(x)] = 0, \end{aligned} \tag{26b}$$

where

$$\theta = \arctan\frac{y}{c}. \tag{27}$$

To derive (26a), the real and imaginary parts of the left-hand side of (24a) are separated, with the result (since $\lim_{y\to\infty}\theta = \pi/2$):

$$\begin{aligned} A\cos(m\pi/2) - B\sin(m\pi/2) &= 0, \\ A\sin(m\pi/2) + B\cos(m\pi/2) &= 0, \end{aligned} \tag{28a}$$

where

$$\begin{aligned} A &= \lim_{y\to\infty}\left\{(c^2+y^2)^{m/2}\left[f_R(y) - \frac{K\cos n\theta}{(c^2+y^2)^{n/2}}\right]\right\}, \\ B &= \lim_{y\to\infty}\left\{(c^2+y^2)^{m/2}\left[f_I(y) + \frac{K\sin n\theta}{(c^2+y^2)^{n/2}}\right]\right\}. \end{aligned} \tag{28b}$$

Therefore $A = B = 0$, and with

$$\lim_{y\to\infty}\frac{(c^2+y^2)^{m/2}}{y^m} = 1, \tag{28c}$$

Eqs. (26a) follow directly. The derivation of (26b) from (24b) is entirely analogous to the one just given, and will therefore be omitted.

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[11]As noted in footnote 5, the left-hand side of (25) might more precisely be written as

$$\mathcal{S}F[\xi(x)] = F(\xi + 0) - F(\xi - 0).$$

The proof of Theorem IX now proceeds as follows[12]. Rewrite (23) as (with $a > 0$)

$$\begin{aligned} F(x) = \frac{1}{\pi}\int_0^a \exp(g_R)(f_R \cos g_I - f_I \sin g_I)\, dy \\ + \frac{1}{\pi}\int_a^\infty \exp(g_R)\left[f_R - \frac{K\cos n\theta}{(c^2+y^2)^{n/2}}\right]\cos g_I\, dy, \\ - \frac{1}{\pi}\int_a^\infty \exp(g_R)\left[f_I + \frac{K\sin n\theta}{(c^2+y^2)^{n/2}}\right]\sin g_I\, dy \\ + \frac{K}{\pi}\int_a^\infty \frac{\exp(g_R)\cos(g_I - n\theta)}{(c^2+y^2)^{n/2}}\, dy. \end{aligned} \tag{29a}$$

The first integral on the right-hand side is a continuous function of x as in *III*, the second and third are continuous functions of x in view of Theorem IV, since (26a) hold and since the first of (26b) implies that $\lim_{y\to\infty} g_R(y, x) = c\xi(x)$. Hence

$$\mathcal{S}F(x) = \frac{K}{\pi}\,\mathcal{S}\int_a^\infty \frac{\exp(g_R)\cos(g_I - n\theta)}{(c^2+y^2)^{n/2}}\, dy. \tag{29b}$$

We rewrite the integral on the right-hand side of (29b) as

$$e^{c\xi}\int_a^\infty \frac{\exp[(g_R - c\xi)] - 1}{(c^2+y^2)^{n/2}}\cos(g_I - n\theta) + e^{c\xi}\int_0^\infty \frac{\cos(g_I - n\theta)}{(c^2+y^2)^{n/2}}\, dy. \tag{30}$$

The first of (26b) implies that a number β can be found such that

$$|\, g_R(y, x) - c\xi(x)\, | < \frac{\epsilon}{y^r} \quad \text{for} \quad y > \beta \tag{31a}$$

and therefore

$$\begin{aligned} \left| \frac{\exp[(g_R - c\xi)] - 1}{(c^2+y^2)^{n/2}} \right| \le \frac{\exp(|\, g_R - c\xi\, |) - 1}{(c^2+y^2)^{n/2}} \le \frac{\exp(\epsilon/y^r) - 1}{(c^2+y^2)^{n/2}} \\ \le \frac{\exp(\epsilon/y^r) - 1}{y^n} = \frac{\epsilon}{y^{n+r}} + \frac{\epsilon^2}{2!y^{n+2r}} + \cdots \end{aligned} \tag{31b}$$

With the introduction of this result in the integrand of the first integral of expression (30), Theorem I applies term-by-term since $n + r > 1$. Thus

$$\mathcal{S}F(x) = \frac{Ke^{c\xi}}{\pi}\,\mathcal{S}\int_a^\infty \frac{\cos(g_I - n\theta)}{(c^2+y^2)^{n/2}}\, dy. \tag{32a}$$

We can apply Theorem V directly to this integral, since for large y,

$$y^m\left[\frac{1}{(c^2+y^2)^{n/2}} - \frac{1}{y^n}\right] = y^{m-n-2}\left[-\frac{nc^2}{2} + \frac{n}{2}\left(\frac{n}{2} + 1\right)\frac{c^4}{y^2} - \cdots\right] \tag{32b}$$

and since a value of $m > 1$ exists for which the limit of this expression as $y \to \infty$ is zero (in fact this limit will be zero as long as $m < n + 2$, where $n + 2 > 1$ since $n > 0$). Hence, according to Theorem V,

$$\mathcal{S}F(x) = \frac{Ke^{c\xi}}{\pi}\,\mathcal{S}\int_a^\infty \frac{\cos(g_I - n\theta)}{y^n}\, dy. \tag{33a}$$

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[12]The similarity between the steps in this proof and those leading to VII is evident.

Thus we may write

$$\begin{aligned} \mathcal{S}F(x) &= \frac{Ke^{c\xi}}{\pi}\,\mathcal{S}\left\{\int_a^\infty \frac{1}{y^n}\,[\cos(g_I - n\theta) - \cos(g_I - n\pi/2)]\,dy \right. \\ &\qquad \left. + \int_a^\infty \frac{1}{y^n}\cos(g_I - n\pi/2)\,dy\right\} \\ &= \frac{2Ke^{c\xi}}{\pi}\,\mathcal{S}\int_a^\infty \sin\frac{n}{2}\left(\theta - \frac{\pi}{2}\right)\sin\left[g_I - \frac{n}{2}\left(\theta + \frac{\pi}{2}\right)\right]\frac{dy}{y^n} \\ &\qquad + \frac{Ke^{c\xi}}{\pi}\,\mathcal{S}\int_a^\infty \frac{\cos(g_I - n\pi/2)}{y^n}\,dy. \end{aligned} \tag{33b}$$

But

$$\theta = \arctan\frac{y}{c} = \frac{\pi}{2} - \frac{c}{y} + \frac{1}{3}\left(\frac{c}{y}\right)^3 - \cdots$$

and thus

$$\left|\theta - \frac{\pi}{2}\right| \le \frac{c}{y} \tag{33c}$$

and furthermore

$$\left|\frac{\sin\frac{n}{2}\left(\theta - \frac{\pi}{2}\right)}{y^n}\right| \le \frac{nc}{2y^{n+1}}. \tag{33d}$$

By Theorem I, therefore, the first integral in the last right-hand side of (33b) is continuous and so

$$\mathcal{S}F(x) = \frac{Ke^{c\xi}}{\pi}\left[\cos\frac{n\pi}{2}\,\mathcal{S}\int_a^\infty \frac{\cos g_I}{y^n}\,dy + \sin\frac{n\pi}{2}\,\mathcal{S}\int_a^\infty \frac{\sin g_I}{y^n}\,dy\right]. \tag{34a}$$

Application of VI directly to each of these two integrals, gives

$$\mathcal{S}F(x) = \frac{Ke^{c\xi}}{\pi}\sin\frac{n\pi}{2}\begin{cases} 0 & \text{for } \xi \ne 0 \\ \left.\begin{array}{lll} 0 & \text{if} & n > 1 \\ \pi & \text{if} & n = 1 \\ \infty & \text{if} & n < 1 \end{array}\right\} & \text{for } \xi = 0 \end{cases} \tag{34b}$$

from which Eqs. (25) immediately follow. The proof of Theorem IX is now complete.

X. It is obvious that a corollary to the preceding theorem, analogous to that of Eqs. (16 a, b), can be written. Condition (24a) is in fact satisfied if, for large p,

$$f(p) = \frac{K}{p^n}\left[1 + 0\left(\frac{1}{p}\right)\right], \qquad n > 0 \tag{35a}$$

and the first of (24b) is satisfied if

$$g(p, x) - p\xi(x) = \frac{K_1}{p} + 0\left(\frac{1}{p^2}\right). \tag{35b}$$

Similarly, the calculation of discontinuities in the derivatives of the function F may be performed by a procedure analogous to that of *VIII*.

4. Illustrative examples. As an illustration of an application of the preceding theory in the case of the Fourier sine and cosine transforms, consider the example of a semi-infinite ($x > 0$) Timoshenko beam, whose end $x = 0$ is subjected to a step moment M_0 for $t > 0$, and is not allowed to displace. The bending moment $M(x, t)$ in the beam is then found by this type of transform to be as follows [4]:

$$\frac{M(x, t)}{M_0} = \frac{2}{\pi} \int_0^\infty \left[\frac{1}{p} + \frac{\lambda_2^2(p^2 - \lambda_1^2\gamma) \cos \lambda_1 t_1 - \lambda_1^2(p^2 - \lambda_2^2\gamma) \cos \lambda_2 t_1}{p^3(\lambda_1^2 - \lambda_2^2)}\right] \sin px_1 \, dp, \tag{36}$$

where

$$2\gamma\lambda_{1,2}^2 = p^2(1 + \gamma) + 1 \mp \{[p^2(\gamma - 1) + 1]^2 + 4p^2\}^{1/2} \tag{36a}$$

$$\gamma = \frac{E}{k'G}; \qquad x_1 = \frac{x}{r}; \qquad t_1 = \frac{c_1 t}{r} \tag{36b}$$

and where r is the radius of gyration of the beam cross-section, c_1 the velocity of propagation of longitudinal waves, k' the shear correction coefficient, and E and G the Young's and shear moduli of the material.

To obtain the discontinuities in the bending moment, rewrite (36) as follows:

$$\begin{aligned} S\left(\frac{M}{M_0}\right) = 2 \Big|_{\text{at } x_1 = 0} &+ \frac{1}{\pi} S \int_0^\infty \frac{\lambda_2^2(p^2 - \lambda_1^2\gamma)}{p^3(\lambda_1^2 - \lambda_2^2)} [\sin (\lambda_1 t_1 + px_1) - \sin (\lambda_1 t_1 - px_1)] \, dp \\ &- \frac{1}{\pi} S \int_0^\infty \frac{\lambda_1^2(p^2 - \lambda_2^2\gamma)}{p^3(\lambda_1^2 - \lambda_1^2)} [\sin (\lambda_2 t_1 + px_1) - \sin (\lambda_2 t_1 - px_1)] \, dp. \end{aligned} \tag{37}$$

For large values of p the following relations are found from Eq. (36a):

$$\lambda_1 = \frac{p}{\sqrt{\gamma}} + 0\left(\frac{1}{p}\right); \qquad \lambda_2 = p + 0\left(\frac{1}{p}\right)$$

$$\frac{\lambda_2^2(p^2 - \lambda_1^2\gamma)}{p^3(\lambda_1^2 - \lambda_2^2)} = 0\left(\frac{1}{p^3}\right); \qquad \frac{\lambda_1^2(p^2 - \lambda_2^2\gamma)}{p^3(\lambda_1^2 - \lambda_2^2)} = \frac{1}{p} + 0\left(\frac{1}{p^3}\right) \tag{37a}$$

$$(\lambda_2 t_1 \pm px_1) - p(t_1 \pm x_1) = 0\left(\frac{1}{p}\right).$$

The first of the integrals on the right-hand side of (37) is thus a continuous function of x_1 and t_1, by virtue of Theorem IV, while the discussion of item VII [and in particular Eqs. (16 a, b)] applies to the second integral with

$$K = 1; \qquad n = 1; \qquad g(p, x_1) = \lambda_2 t_1 \pm px_1; \qquad \xi(x_1) = t_1 \pm x_1. \tag{37b}$$

Hence use of Eqs. (15) gives immediately

$$S\left(\frac{M}{M_0}\right) = \begin{cases} 2 \, |_{\text{at } x_1 = 0} - 1 \, |_{\text{at } t_1 + x_1 = 0} + 1 \, |_{\text{at } t_1 - x_1 = 0} \\ 0 \quad \text{everywhere else} \end{cases}. \tag{38}$$

The discontinuity of an amount 2 at $x_1 = 0$ represents the applied bending moment[13], The points for which $t_1 + x_1 = 0$ are not within the physical structure, and the corresponding term in (38) may thus be disregarded. The last term indicates that the bending

[13] It is actually equal to twice the applied moment because of the manner in which the saltus was defined in Eq. (4). Here the quantity $M(0-, t_1)$ has no meaning, and the true discontinuity is $[M(0+, t_1) - M(0, t_1)] = M_0$.

moment discontinuity is propagated along the beam, without change in magnitude,[14] at a constant velocity c_1.

To illustrate the use of the preceding theory in the case of the Laplace transforms, consider again the beam of the preceding example, but subjected at $x = 0$ to a step-variation of transverse velocity of magnitude v_0 for $t > 0$, and with zero bending moment there. The bending moment $M(x, t)$ at any point of the beam is then given by [2]:

$$\frac{Mrc_1}{EIv_0} = \frac{1}{2\pi i(\gamma - 1)} \int_{c-i\infty}^{c+i\infty} \frac{\exp (pt_1)}{p[p^2 - 4/(\gamma - 1)^2]^{1/2}} [\exp (-\mu_2 x_1) - \exp (-\mu_1 x_1)]\, dp, \tag{39}$$

where I is the moment of inertia of the beam cross-section,

$$\mu_{1,2}^2 = \frac{(\gamma + 1)p}{2} \left\{p \pm \left(\frac{\gamma - 1}{\gamma + 1}\right)\left[p^2 - \left(\frac{2}{\gamma - 1}\right)^2\right]^{1/2}\right\} \tag{39a}$$

and the other symbols have been previously defined. Separation of real and imaginary parts, and use of symmetry properties, would put (39) in the form (23). For large values of p we have:

$$\begin{gathered} \mu_1 = p(\gamma)^{1/2} + 0\left(\frac{1}{p}\right); \qquad \mu_2 = p + 0\left(\frac{1}{p}\right) \\ \frac{1}{p[p^2 - 4/(\gamma - 1)^2]^{1/2}} = \frac{1}{p^2} + 0\left(\frac{1}{p^4}\right) \\ (pt_1 - \mu_2 x_1) - p(t_1 - x_1) = 0\left(\frac{1}{p}\right); \qquad (pt_1 - \mu_1 x_1) - p[t - x_1(\gamma)^{1/2}] = 0\left(\frac{1}{p}\right) \end{gathered} \tag{40}$$

and therefore the bending moment is a continuous function of x and t. Not so its derivative with respect to x, for we may write

$$\begin{aligned} \mathrm{S}\, \frac{\partial}{\partial x_1} \left(\frac{Mrc_1}{EIv_0}\right) = \frac{1}{2\pi i(\gamma - 1)} \Biggl\{ \mathrm{S} \int_{c-i\infty}^{c+i\infty} \frac{\mu_1 \exp (pt_1 - \mu_1 x_1)}{p[p^2 - 4/(\gamma - 1)^2]^{1/2}}\, dp \\ - \mathrm{S} \int_{c-i\infty}^{c+i\infty} \frac{\mu_2 \exp (pt_1 - \mu_2 x_1)}{p[p^2 - 4/(\gamma - 1)^2]^{1/2}}\, dp \Biggr\}, \end{aligned} \tag{41}$$

where the integrals on the right-hand side are discontinuous only at isolated points. In fact, for large p,

$$\begin{aligned} \frac{\mu_1}{p[p^2 - 4/(\gamma - 1)^2]^{1/2}} &= \frac{(\gamma)^{1/2}}{p} + 0\left(\frac{1}{p^3}\right), \\ \frac{\mu_2}{p[p^2 - 4/(\gamma - 1)^2]^{1/2}} &= \frac{1}{p} + 0\left(\frac{1}{p^3}\right), \end{aligned} \tag{41a}$$

and thus with

$$K = (\gamma)^{1/2}; \qquad n = 1; \qquad g(p, x_1) = pt_1 - \mu_1 x_1; \qquad \xi(x_1) = t_1 - x_1(\gamma)^{1/2}. \tag{42a}$$

[14] A remark concerning the sign of this discontinuity may be useful. If M is considered as a function of x_1, with t_1 fixed, then $M = 0$ for $x_1 > t_1$ and $M = 1$ for $x_1 = t_1 - 0$; hence

$$M(x_1 + 0) - M(x_1 - 0) = -1.$$

This differs in sign from the result given in Eq. (38); the reason for this is that in Eq. (38) M was regarded as a function of ξ rather than of x_1. In that case then

$$M(\xi + 0) - M(\xi - 0) = M(t_1 - x_1 + 0) - M(t_1 - x_1 - 0) = +1.$$

The same situation arises in the illustrative example which follows.

in the first integral of (41) and with

$$K = 1; \quad n = 1; \quad g(p, x_1) = pt_1 - \mu_2 x_1 ; \quad \xi(x_1) = t_1 - x_1 \tag{42b}$$

in the second, Eqs. (35 a, b) apply and give, with Theorem IX,

$$\mathcal{S}\frac{\partial}{\partial x}\left(\frac{Mrc_1}{EIv_0}\right) = \frac{(\gamma)^{1/2}}{\gamma - 1}\Bigg|_{\text{at } t_1 - x_1(\gamma)^{1/2} = 0} - \frac{1}{\gamma - 1}\Bigg|_{\text{at } t_1 - x_1 = 0}. \tag{42}$$

Hence[15] the bending moment distribution, though continuous, has a "kink" (namely a discontinuity in its slope) traveling with a velocity c_1, of magnitude $[1/(\gamma - 1)]$, and another one, traveling with a velocity $c_1/(\gamma)^{1/2}$ of magnitude $[(\gamma)^{1/2}/(\gamma - 1)]$.

As a last example, consider the calculation of discontinuities in the derivative of a function for which a divergent integral is obtained when the processes of integration and differentiation are interchanged as in Eq. (17). An example of this is given by

$$F(x) = \int_0^\infty \frac{\sin px}{p + a}\, dp \tag{43}$$

since clearly the integral

$$\int_0^\infty \frac{p \cos px}{p + a}\, dp$$

does not exist. We can however write

$$\mathcal{S}\frac{dF}{dx} = \mathcal{S}\frac{d}{dx}\int_0^\infty \left(\frac{1}{p + a} - \frac{1}{p}\right) \sin px\, dp = -a\mathcal{S}\int_0^\infty \frac{\cos px}{p + a}\, dp = 0 \tag{44}$$

since the function

$$\frac{2}{\pi}\int_0^\infty \frac{\sin px}{p}\, dp = \begin{cases} 1 & \text{for} \quad x > 0 \\ -1 & \text{for} \quad x < 0 \end{cases}$$

obviously has no saltus in its derivative.

These examples might be concluded by a mention of two other methods which have been used to determine the magnitude and location of discontinuities in problems of this type. One of these is of course the method of characteristics (for an application to Timoshenko beam problems see for example [5]); the other is a variational method, in which these quantities are found by satisfying the governing differential equations and certain dynamical and kinematical conditions at the point of discontinuity [6].

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[15]The signs in Eq. (42) are listed with the left-hand side considered a function of ξ rather than of x_1; see footnote 14.