The question of uniqueness concerns the possibility of two or more solutions to a boundary value problem of the type just described.

Suppose now that two solutions exist. Let $\Delta\sigma_{ij}(x, t)$, $\Delta\epsilon_{ij}(x, t)$ and Δv_i denote the differences of these solutions. We then have—in view of some of the regularity requirements stated above—from the divergence theorem [5] and the boundary conditions (15)

$$\int_D \frac{\partial}{\partial x_i} (\Delta\sigma_{ij}\, \Delta v_i)\, dV = \int_B \Delta\sigma_{ij}\, \Delta v_i n_i\, dS = \int_B \Delta T_i\, \Delta v_i\, dS = 0. \tag{16}$$

On the other hand with the use of (3) and (4), (16) reduces to

$$\int_D \Delta\sigma_{ij}\, \Delta\dot{\epsilon}_{ij}\, dV = 0.$$

Now by integrating (16) with respect to t and noting that $\Delta\sigma_{ij}$ and $\Delta\dot{\epsilon}_{ij}$ also satisfy the constitutive law (2) we obtain

$$\int_0^T dt \int_D \Delta\sigma_{ij}\, \Delta\dot{\epsilon}_{ij}\, dV = \int_D \left(\int_0^T \Delta\sigma_{ij}\, \Delta\dot{\epsilon}_{ij}\, dt\right) dV = \int_D W[x, \underset{0}{\overset{T}{\Delta\epsilon_{ij}}}]\, dV = 0. \tag{17}$$

If W is positive definite then (17) demands that

$$W[x, \underset{0}{\overset{T}{\Delta\epsilon_{ij}}}] = 0$$

in D and therefore $\Delta\dot{\epsilon}_{ij}(x, t)$ and hence $\Delta\epsilon_{ij}(x, t)$ and $\Delta\sigma_{ij}(x, t)$ must vanish identically in the time interval $[0, T]$ everywhere in D. The last conclusion implies that there cannot exist two distinct stress and strain fields satisfying (2), (3), (4) and (5) and (15).

Acknowledgment. The authors are indebted to Professors M. Rosenblatt and E. Sternberg for many helpful discussions and to Professor M. Rosenblatt for calling their attention to Bochner's theorem.

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DISTORTION OF BOUSSINESQ FIELD BY CIRCULAR HOLE*

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Introduction. The classical Boussinesq solution to the problem of a concentrated load acting on the straight boundary of a semi-infinite plate is basic to a number of problems in the plane theory of elasticity. Barjansky [1] modified the Boussinesq problem and analyzed the effects of a circular hole in the plate. In the following paper the latter problem has been restated and some corrections affecting the results have been made.**

*Received May 4, 1961.

**Calculations are shown in Appendix.

Statement of the problem and general procedure. Consider a semi-infinite thin plate containing a circular hole and loaded by a concentrated normal force P (case N), Fig. 1a, or a concentrated tangential force P (case T), Fig. 1b. The subscript 1 will refer to the case N and subscript 2 to the case T.

To the Boussinesq stress function, ϕ, the stress function χ is added such that stress function $\Phi = \phi + \chi$ will satisfy the boundary conditions around the circular hole, i.e., no normal and shearing stresses there. Besides, the function χ must be chosen so that it renders no stresses at infinity and results in zero normal and shearing stresses on the straight boundary. The known Boussinesq solutions are (see Fig. 1a and 1b)

$$\begin{aligned} \phi_1 &= -(P/\pi)(y - y_0)\tan^{-1}[(y - y_0)/x], \\ \phi_2 &= -(P/\pi)x\tan^{-1}[(y - y_0)/x]. \end{aligned} \tag{1}$$

The problem is solved in bipolar coordinates:

$$x = -J\sinh\xi; \qquad y = J\sin\eta; \qquad J = a(\cosh\xi - \cos\eta)^{-1}. \tag{2}$$

For the case considered here $\xi \leqq 0$. The circular boundary is specified by setting $\xi = \xi_0 =$ const, and the straight boundary $\xi = 0$.

Denote:

$$\beta = \tan^{-1}(y_0/a); \qquad p = \sin\beta\cosh\xi; \qquad q = \cos\beta\sinh\xi; \qquad \psi = \eta + \beta. \tag{3}$$

Thus (1) can be written in bipolar coordinates as

$$\phi_2/J = (P/\pi)\sinh\xi\tan^{-1}[(p - \sin\psi)/q]; \qquad \phi_1/J = (\phi_2/J)[(p - \sin\psi)/q]. \tag{4}$$

General form of the stress function χ in bipolar coordinates is known, see [2], and the stress function χ satisfying the above imposed conditions is

$$\left.\begin{aligned} \chi/J &= B\xi\cosh\xi - [B(\xi - \sinh\xi\cosh\xi) + 2F\sinh^2\xi]\cos\eta + (G'\cosh 2\xi + F')\sin\eta \\ &\quad + \sum_{k=2}^{\infty} 2\{[E_k\alpha_k(\xi) + F_k\beta_k(\xi)]\cos k\eta + [E_k'\alpha_k(\xi) + F_k'\beta_k(\xi)]\sin k\eta\}/(k-1), \\ &\text{where } \alpha_k(\xi) = (k-1)\sinh\xi\sinh k\xi; \qquad \beta_k(\xi) = k\sinh\xi\cos k\xi - \cosh\xi\sinh k\xi. \end{aligned}\right\} \tag{5}$$

The unknown coefficients in (5) are determined from the boundary conditions around the circular hole. To this end the Boussinesq stress functions are expanded into Fourier series in η.

Fourier representation of functions ϕ. Starting with ϕ_2, we have

$$\phi_2/J = R_0/2 + \sum_{k=1}^{\infty}(R_k\cos k\eta + S_k\sin k\eta),$$

where

$$\left.\begin{aligned} R_0/2 &= (P\sinh\xi/\pi)\tan^{-1}(\tan\beta\,\mathrm{cth}\,\xi) \\ &\quad + \sinh\xi\tan^{-1}[\sin 2\beta/(\cos 2\beta - e^{-2\xi})]\}, \\ R_k &= -(P\sinh\xi/\pi)e^{k\xi}(-1)^k\sin k\beta/k, \\ S_k &= -(P\sinh\xi/\pi)[1 - (-1)^k\cos k\beta]/k. \end{aligned}\right\}. \tag{6}$$

see [1], p. 21)

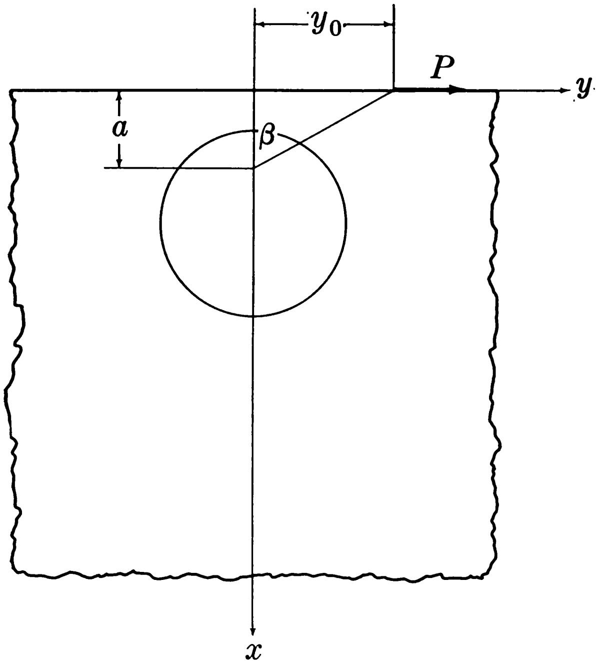


FIG. 1a

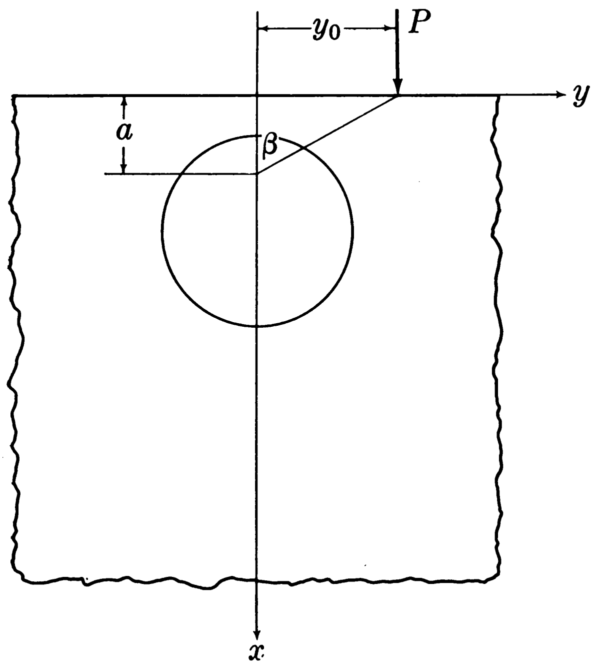


FIG. 1b

The Fourier coefficients of ϕ_1/J are found from (6) by multiplying ϕ_2/J by $(p - \sin\psi)/q$, see (3).

Case T. Tangential concentrated force P acting in the straight boundary, Fig. 1b. The required stress function χ_2 has the form given in (5). Thus

$$\Phi_2/J = \phi_2/J + \chi_2/J,$$

the unknown coefficients in χ_2 are found from the boundary conditions around the circular hole ξ_0. For this case:

$$F = (P/\pi)\{(1 + 4\sinh\xi_0)\sin 2\beta/d_1 + e^{2\xi_0}/\sinh\xi_0 - \cosh\xi_0\tan^{-1}(\sin 2\beta/d_1^*)\}$$

$$B = (P/2\pi\sinh^2\xi_0)\{\sinh 2\xi_0(1 + 4\sinh\xi_0)\sin 2\beta/d_1 - \cosh\xi_0\sinh 2\xi_0\tan^{-1}(\sin 2\beta/d_1^*) + e^{2\xi_0}(2\cosh\xi_0 - 1)\sin\beta$$

$$G' = -(P/\pi)(1 + \cos\beta)/(1 - 4e^{-4\xi_0})\}$$

$$F' = 0$$

where

$$4(\cosh 2\xi_0 - \cos 2\beta) = d_1\,; \qquad \cos 2\beta - e^{-2\xi_0} = d_1^*. \tag{a}$$

The unknown coefficients for $k \geq 2$ are:

$$E_k = [-P(-1)^k\sin k\beta(k^2\sinh^2\xi_0 + k\sinh\xi_0\cosh\xi_0 - e^{-k\xi_0}\sinh k\xi_0)]/k\,D_k^*$$

$$F_k = -P(-1)^k(k - 1)\sin k\beta\sinh^2\xi_0/D_k^*$$

$$E_k' = P[1 - (-1)^k\cos k\beta](k^2\sinh^2\xi_0 + k\sinh\xi_0\cosh\xi_0 - e^{k\xi_0}\sinh k\xi_0)/k\,D_k^*$$

$$F_k' = P(k - 1)[1 - (-1)^k\cos k\beta]\sinh^2\xi_0/D_k^*$$

$$D_k^* = 2\pi(\sinh^2 k\xi_0 + k^2\sinh^2\xi_0).$$

(see [1], p. 25)

Case N. Normal concentrated force P acting on the straight boundary, Fig. 1a. In this case the Boussinesq stress function ϕ_1 is represented as

$$\phi_1/J = T_0/2 + \sum_{k=1}^{\infty}(T_k\cos k\eta + U_k\sin k\eta),$$

where

$$T_0/2 = (P/\pi)\{\tan\beta\cosh\xi[\tan^{-1}(\tan\beta\operatorname{cth}\xi) + \sinh\xi\tan^{-1}[\sin 2\beta/(\cos 2\beta - e^{-2\xi})]] - e^{\xi}(1 + \cos\beta)/2\cos\beta\},$$

$$T_1 = (P/\pi)\tan\beta\{e^{\xi}\sin\beta\cosh\xi - \tan^{-1}(\tan\beta\operatorname{cth}\xi) - \sinh\xi\tan^{-1}[\sin 2\beta/(\cos 2\beta - e^{-2\xi})]\},$$

$$U_1 = (P/\pi)\{\tfrac{1}{2}\tan\beta[1 + (1 + e^{2\xi})\cos\beta] - \tan^{-1}(\tan\beta\operatorname{cth}\xi) - \sinh\xi\tan^{-1}[\sin 2\beta/(\cos 2\beta - e^{-2\xi})]\},$$

$$T_k = -(Pe^{k\xi}/\pi)\{A_k(\xi) + B_k(\xi)[1 + (-1)^k(\cos k\beta)/\cos\beta]\},$$

$$U_k = -(Pe^{k\xi}/\pi)\{A_k(\xi) - B_k(\xi)[(\tan\beta)/k + (-1)^k(\sin k\beta)/\cos\beta]\},$$

where

$$A_k(\xi) = (-1)^k \tan \beta \sin k\beta \cosh \xi/k,$$
$$B_k(\xi) = (k \sinh \xi - \cosh \xi)/(k^2 - 1).$$

Moreover

$$F = (P/2\pi \sinh^2 \xi_0)\{e^{\xi_0} \tan \beta \sin \beta \cosh \xi_0 - (1 + \cos \beta)/2 \cos \beta - \tan \beta \sinh \xi_0 \cosh \xi_0 \cdot [\sin 2\beta(1 + 4 \sinh \xi_0)/4(\cos 2\beta - \cosh 2\xi_0) + \cosh \xi_0 \tan^{-1} (\sin 2\beta/d_1^*)]\},$$

$$B = (1/2 \sinh^2 \xi_0)\{4F \cosh \xi_0 \sinh \xi_0 + (P/\pi) \tan \beta \tan^{-1} (\sin 2\beta/d_1^*) - \sin 2\beta(1 - 4 \sinh \xi_0)/d_1 - e^{-2\xi_0} \sin \beta]\},$$

$$G' = (P/2\pi \sinh^2 \xi_0)\{e^{2\xi_0} \sin \beta - \sin 2\beta(1 + 4 \sinh \xi_0)/d_1 + \cosh \xi_0 \tan^{-1} (\sin 2\beta/d_1^*)\},$$

$$F' = 0,$$

where d_1 and d_1^* are given by (a).

The unknown coefficients E_k, F_k, E_k' and F_k' for $k \geqq 2$ are:

$$E_k = -(P/2\pi)\{c_k \sin k\beta + k\, d_k \sinh \xi_0\}/D_k ,$$
$$F_k = -(P/2\pi)\{e_k \sin k\beta + d_k f_k (k + 1)^{-1}\}/D_k ,$$
$$E_k' = (P/2\pi)\{c_k \cos k\beta - k g_k \sinh^2 \xi_0\}/D_k ,$$
$$F_k' = (P/2\pi)\{e_k \cos k\beta - (k + 1)^{-1} g_k f_k\}/D_k ,$$

where

$$c_k = (-1)^k \tan \beta(k \sinh k\xi_0 \cosh \xi_0 + \sinh^2 \xi_0 - e^{k\xi_0} \sinh k\xi_0),$$
$$d_k = [1 - (-1)^k \cos k\beta/\cos \beta],$$
$$e_k = (-1)^k(k - 1)k^{-1} \tan \beta(k \sinh \xi_0 \cosh \xi_0 + e^{k\xi_0} \sinh k\xi_0),$$
$$f_k = (k^2 \sinh^2 \xi_0 - k \sinh \xi_0 - e^{k\xi_0} \sinh k\xi_0),$$
$$g_k = [k^{-1} \tan \beta + (-1)^k \sin k\beta/\cos \beta],$$
$$D_k = (\sinh^2 k\xi_0 - k^2 \sinh^2 \xi_0).$$

Appendix

Case k = 0. Calculation of R_0, for $\beta \neq 0$, (see [1], p. 28)

$$R_0 = (P \sinh \xi/\pi^2) \int_{-\pi}^{\pi} \tan^{-1} [(p - \sin \psi)/q] e^{ik\psi} = (p/\pi^2) \sinh \xi \cdot I_0 .$$

Integrating I_0 by parts we get

$$I_0 = [\psi \tan^{-1} \{(p - \sin \psi)/q\}]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} q\psi \cos \psi \, d\psi/(p - \sin \psi)^2 = 2\pi \tan^{-1} (p/q) \cdot J_0 .$$

J_0 can be represented as

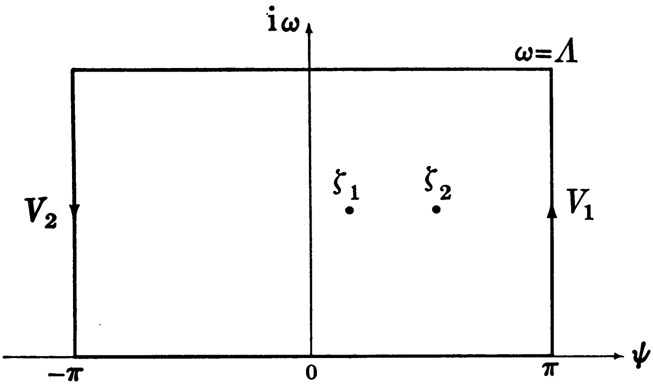
$$J_0 = -(2 \cos 2\beta)^{-1} \int_{-\pi}^{\pi} \psi[(e^{i\psi} - e^{\xi+i\beta})^{-1} - (e^{i\psi} - e^{-\xi+i\beta})^{-1} - (e^{i\psi} + e^{\xi-i\beta})^{-1} + (e^{i\psi} + e^{-\xi-i\beta})^{-1}]e^{i\psi}\, d\psi.$$

A typical integral is:

$$\int_{-\pi}^{\pi} \psi e^{i\psi}(e^{i\psi} - c_k^*)^{-1}\, d\psi.$$

To evaluate these integrals, a complex variable ζ is introduced

$$\zeta = \psi + i\omega$$



and the integration proceeds along the contour, C, shown in the figure. Single poles $\zeta_1 = \beta - i\xi$ and $\zeta_2 = \pi - \beta - i\xi$ of the first and second terms are located within the contour of integration as shown in the figure, and the second and fourth term have single poles outside of the contour of integration. Thus

$$\int_C = \int_{V_1} + \int_{V_2} + \int_{\omega=\Lambda} + \int_{-\pi}^{\pi}$$

is equal to $2\pi i$ Res ($\zeta = \beta - i\xi$) for the first integral; is equal to $2\pi i$ Res ($\zeta = \pi - \beta - i\xi$) for the third integral and zero for the second and fourth. Here V_1 and V_2 are the vertical segments of the integration contour. On V_1, $\zeta = \pi + i\omega$ and on V_2, $\zeta = -\pi + i\omega$. It is a simple matter to demonstrate that the integrals for $\omega = \Lambda \to \infty$ tend to zero. Obviously the last integrals represent parts of J_0. Now we proceed to evaluate the integrals along V_1 and V_2.

$$\int_{V_1} - \int_{V_2} = \int_0^{\Lambda} (\pi + i\omega)e^{i(\pi+i\omega)}[e^{i(\pi+i\omega)} - c_k^*]^{-1}\, d(i\omega)$$
$$- \int_0^{\Lambda} (-\pi + i\omega)e^{i(-\pi+i\omega)}[e^{i(-\pi+i\omega)} - c_k^*]^{-1}\, d(i\omega)$$
$$= 2\pi i \int_0^{\Lambda} e^{-\omega}(e^{-\omega} + c_k^*)^{-1}\, d\omega = 2\pi i[\ln (e^{-\omega} + c_k^*)]_0^{\Lambda}.$$

Letting $\Lambda \to \infty$, we get

$$\lim_{\Lambda\to\infty} 2\pi i[\ln (e^{-\omega} + c_k^*)]_0^\Lambda = 2\pi i[\ln c_k^* - \ln (1 + c_k^*)] = 2\pi i \ln [c_k^*/(1 + c_k^*)].$$

Summing up for all $k(k = 1, 2, 3, 4)$ we get

$$\sum_{k=1}^{4} \int_C = 2 \cos \beta \cdot J_0 + 2\pi i\{2i\beta + 2i \tan^{-1} [\sin 2\beta/(\cos 2\beta - e^{-2\xi})]\} = 2\pi i(2i\beta),$$

where Res (ζ_1) + Res $(\zeta_2) = 2i\beta$. Thus

$$J_0 = (\pi/\cos \beta) \tan^{-1} [\sin 2\beta/(\cos 2\beta - e^{-2\xi})]$$

and consequently

$$I_0 = 2\pi\{\tan^{-1} (\tan \beta \coth \xi) + \sinh \xi \tan^{-1} [\sin 2\beta/(\cos 2\beta - e^{-2\xi})]\}$$

and finally

$$R_0/2 = (P \sinh \xi/\pi)\{\tan^{-1} (\tan \beta \operatorname{cth} \xi) + \sinh \xi \tan^{-1} [\sin 2\beta/(\cos 2\beta - e^{-2\xi})]\}.$$

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