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ON UNIQUENESS IN LINEAR VISCOELASTICITY*

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Summary. It is shown that solutions of a class of boundary value problems in linear vicoelasticity are unique, if the relaxation moduli in shear and compression are steadily decreasing functions of time which are convex from below and tend to non-negative constant asymptotic values.

1. **Introduction.** Consider isothermal deformations of a linear isotropic viscoelastic solid. Let $\sigma_{ij}(x, t)$ and $\epsilon_{ij}(x, t)$ denote the components of the stress and infinitesimal strain tensors respectively in the rectangular cartesian coordinates x_i. Here, as in the sequel, the single argument x stands for the triplet of coordinates (x_1, x_2, x_3), while t denotes the time. With a view of stating constitutive laws governing the mechanical behavior in a convenient form we introduce the deviatoric components of stress and strain

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\,\delta_{ij}\sigma_{kk}, \qquad e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\,\delta_{ij}\epsilon_{kk}, \tag{1}$$

where δ_{ij} denotes the Kronecker-delta. We shall be concerned with the following integral representation of the mechanical behavior [1]**

$$\begin{aligned} s_{ij}(x, t) &= \int_0^t G_1(t - \tau)\,\frac{\partial}{\partial \tau}\,e_{ij}(x, \tau)\,d\tau, \\ \sigma_{kk}(x, t) &= \int_0^t G_2(t - \tau)\,\frac{\partial}{\partial \tau}\,\epsilon_{kk}(x, \tau)\,d\tau, \end{aligned} \tag{2}$$

where $G_1(t)$ and $G_2(t)$ are the relaxation moduli in pure shear and isotropic compression, respectively. This representation contains the tacit assumption that the solid is in the unstressed and unstrained virgin state for $t < 0$. Note also that G_1 and G_2 need only be defined for non-negative values of their arguments.

In view of the relative scarcity and incompleteness of experimental information concerning the moduli $G_1(t)$ and $G_2(t)$ it is important to know what restrictions can be imposed upon G_1 and G_2 on physical and mathematical grounds.

One set of such restrictions arises from the considerations of uniqueness in boundary value problems of the quasi-static linear theory of viscoelasticity. The complete system of field equations for such a boundary value problem consists of the equations of equilibrium

$$\frac{\partial}{\partial x_j}\,\sigma_{ij}(x, t) = 0, \tag{3}$$

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**Numbers in square brackets refer to the Bibliography at the end of the paper.

the displacement-strain relations

$$\epsilon_{ij}(x, t) = \frac{1}{2}\left[\frac{\partial}{\partial x_j} u_i(x, t) + \frac{\partial}{\partial x_i} u_j(x, t)\right], \tag{4}$$

and the constitutive law (2).

To these field equations, which must hold throughout the *finite regular* region D occupied by the solid we adjoin the following initial and boundary conditions.

We assume that the body is initially undisturbed, so that

$$u_i(x, t) = \sigma_{ij}(x, t) = 0 \quad \text{in} \quad D \quad \text{for} \quad t < 0. \tag{5}$$

For $t \geq 0$ surface tractions $T_i(x, t)$ are prescribed on the part S_F of the boundary B of D and displacements $u_i(x, t)$ on the remaining part of the boundary. Note that S_F may be time dependent.

If we demand that the solutions of the class of boundary value problems just described be unique*, then $G_1(t)$ and $G_2(t)$ cannot be arbitrary but must meet certain requirements.

The present paper is concerned with the elucidation of these requirements. As will be seen in the next section these requirements on G_1 and G_2 (Theorems 1-3) are not surprising and they may even be intuitively plausible. However it is hoped that the establishment of these requirements with the help of well-known results of analysis may constitute a starting point towards the construction of variational principles governing deformations of viscoelastic solids for finite intervals of time.

2. Work density. We consider an arbitrary deformation of a volume element at x_i and evaluate the work done by the stresses (per unit volume of the solid) during the time interval $[0, T]$

$$W[x, \overset{T}{\underset{0}{\epsilon_{ij}}}] = \int_0^T \sigma_{ij}(x, t) \frac{\partial}{\partial t} \epsilon_{ij}(x, t)\, dt = \int_0^T \left[s_{ij} \frac{\partial}{\partial t} e_{ij} + \tfrac{1}{3}\sigma_{kk} \frac{\partial}{\partial t} \epsilon_{pp}\right] dt. \tag{6}$$

Now by substituting for s_{ij} and σ_{kk} from (2) and extending the range of definition of G_1 and G_2, by the definition

$$G_\alpha(t) = G_\alpha(-t) \qquad (\alpha = 1, 2) \tag{7}$$

(6) may be written in the following form

$$2W[x, \overset{T}{\underset{0}{\epsilon_{ij}}}] = \iint_\Omega G_1(t - \tau) \frac{\partial}{\partial \tau} e_{ij}(x, \tau) \frac{\partial}{\partial t} e_{ij}(x, t)\, d\tau\, dt$$

$$+ \frac{1}{3} \iint_\Omega G_2(t - \tau) \frac{\partial}{\partial \tau} \epsilon_{kk}(x, \tau) \frac{\partial}{\partial t} \epsilon_{pp}(x, t)\, d\tau\, dt, \tag{8}$$

where Ω is the square domain in the (t, τ) plane defined by the inequalities

$$0 \leq t, \qquad \tau \leq T.$$

The main purpose of the present paper is to establish the restrictions to be imposed upon G_1 and G_2 in order to ensure positive definiteness of W. The importance of the

*As will be seen in the last section we shall also demand that boundary value problems considered admit solutions satisfying certain regularity requirements.

positive definiteness of W has long been recognized in the theory of elasticity. Recently Drucker [2] has emphasized the significance of definiteness of W or other similar forms in the mechanics of continua. As will be seen in the next section, the positive definiteness of W results in the uniqueness of the solutions of the class of boundary value problems considered in this paper.

In discussing positive definiteness of W we first note that the right-hand side of (8) is the sum of terms of the type

$$w = \iint_{\Omega} G(t - \tau)y(t)y(\tau)\,dt\,d\tau, \tag{9}$$

where G stands for either G_1 or G_2 and $y(t)$ for $\partial e_{ij}/\partial t$ or $\partial \epsilon_{kk}/\partial t$. Moreover if each of these terms is positive definite so is the functional (8). This leads us to examine the conditions for the positive definiteness of the functional defined in (9). Such requirements are known, however, from the theory of Fourier Integrals. In fact, Bochner's theorem [3] which plays an important part in the theory of probability provides necessary and sufficient conditions for $G(t)$ which ensure the non-negative definiteness of (9).

For the purposes of the present study it may suffice to give the following simplified version of Bochner's Theorem which deals only with the Riemann integrals and aims only at sufficient conditions.

Theorem 1. Let the even function $G(t)$ be continuous and piecewise smooth. If $\int_0^\infty \mid G(t) \mid dt$ exists and $\bar{G}(u)$, the Fourier transform of $G(t)$, is positive for all real values of u then w is positive definite, i.e.

$$w = \iint_{\Omega} G(t - \tau)y(t)y(\tau)\,dt\,d\tau > 0 \tag{10}$$

for any piecewise continuous function $y(t)$ which does not vanish identically in the interval $[0, T]$.

It may be useful to sketch the proof of the theorem. From (9) and the Fourier inversion theorem we have

$$w = \frac{1}{(2\pi)^{1/2}} \iint_{\Omega} \left(\int_{-\infty}^{+\infty} \exp\,[-i(t - \tau)u]\bar{G}(u)\,du \right) y(t)y(\tau)\,dt\,d\tau. \tag{11}$$

Under the conditions of the theorem the improper integral in (11) converges uniformly [4, p. 13] so that the order of integration in (11) may be interchanged to obtain

$$w = \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{+\infty} \bar{G}(u) \left| \int_0^T \exp\,(-itu)y(t)\,dt \right|^2 du. \tag{12}$$

Now since $y(t)$ is not identically zero in $[0, T]$ there exists [4, p. 164] at least one subinterval of $-\infty < u < +\infty$ in which

$$\left| \int_0^T \exp\,(-itu)y(t)\,dt \right| > 0,$$

and therefore since $\bar{G}(u) > 0$ for $-\infty < u < +\infty$ the inequality (6) follows.

Note that relaxation moduli of the form

$$G(t) = \sum_i^N C_i \exp\left(-\frac{t}{\tau_i}\right) \qquad (t \geq 0), \tag{13}$$

where $C_i > 0$, $\tau_i > 0$ satisfy the conditions of the theorem.

On the other hand the restrictions of Theorem 1 on $G(t)$ can be relaxed in two important respects as will be seen from the following theorems.

Theorem 2. If the even function $G(t)$ is bounded and decreases steadily to 0 as $t \to \infty$ and *if it is convex from below* for $t > 0$ then w is positive definite.

The proof of the theorem follows from a well-known lemma [4, p. 170] and may be carried out along the lines of the previous proof.

Theorem 3. If

$$\mathcal{G}(t) = C + G(t), \tag{14}$$

where C is a non-negative constant and $G(t)$ satisfies the requirements of Theorems 1 or 2, then

$$w = \int_0^T \mathcal{G}(t - \tau) y(t) y(\tau)\, dt\, d\tau$$

is positive definite. The proof follows from the observation

$$\iint_\Omega y(t) y(\tau)\, dt\, d\tau = \left[\int_0^T y(t)\, dt \right]^2 > 0$$

and Theorems 1 and 2.

Returning to viscoelastic materials we can now make the following statement: If the relaxation moduli $G_1(t)$ and $G_2(t)$ satisfy the requirements of Theorem 3 then W is positive definite.

Here we shall not discuss how the creep compliances and—in case (2) admits a differential operator representation—the differential operators are effected by these requirements on the relaxation moduli. However, it may be useful to summarize some of the previous results in the following physical terms:

If for a linear viscoelastic material the relaxation moduli in shear and compression versus time curves decrease steadily with time to non-negative constant asymptotic values and if these curves are convex from below then positive work must be done in order to deform such a material from the unstressed and unstrained state.

3. Uniqueness. We now show, following the basic ideas of Drucker [2], how the positive definiteness of W is related to uniqueness.

In boundary value problems defined in the introduction it is required to determine $\sigma_{ij}(x, t)$, $\epsilon_{ij}(x, t)$ and $u_i(x, t)$ in D satisfying (2), (3) and (4) and subject to the initial conditions (5) and the boundary conditions

$$\left.\begin{aligned} \sigma_{ij} n_j = T_i = f_i(x, t) \quad &\text{on} \quad S_F \\ u_i = h_i(x, t) \quad &\text{on} \quad B - S_F \end{aligned}\right\} \quad \text{for} \quad t \geq 0, \tag{15}$$

where n_j is the unit outward normal of B and $f_i(x, t)$ and $h_i(x, t)$ are given functions of time and position on B.

Here we shall be interested in boundary value problems of this type which admit solutions satisfying the following regularity requirements.

At any time $\sigma_{ij}(x, t)$ and $v_i = \partial u_i / \partial t$ are continuous in D with piecewise continuous partial derivatives with respect to x_i. Moreover $\dot{\epsilon}_{ij} = \partial \epsilon_{ij}(x, t)/\partial t$ are piecewise continuous in t for all times. The last condition implies, in view of (2), that $\sigma_{ij}(x, t)$ and $T_i(x, t)$ are continuous in t for all times.

The question of uniqueness concerns the possibility of two or more solutions to a boundary value problem of the type just described.

Suppose now that two solutions exist. Let $\Delta\sigma_{ij}(x, t)$, $\Delta\epsilon_{ij}(x, t)$ and Δv_i denote the differences of these solutions. We then have—in view of some of the regularity requirements stated above—from the divergence theorem [5] and the boundary conditions (15)

$$\int_D \frac{\partial}{\partial x_j} (\Delta\sigma_{ij}\, \Delta v_i)\, dV = \int_B \Delta\sigma_{ij}\, \Delta v_i n_j\, dS = \int_B \Delta T_i\, \Delta v_i\, dS = 0. \tag{16}$$

On the other hand with the use of (3) and (4), (16) reduces to

$$\int_D \Delta\sigma_{ij}\, \Delta\dot{\epsilon}_{ij}\, dV = 0.$$

Now by integrating (16) with respect to t and noting that $\Delta\sigma_{ij}$ and $\Delta\dot{\epsilon}_{ij}$ also satisfy the constitutive law (2) we obtain

$$\int_0^T dt \int_D \Delta\sigma_{ij}\, \Delta\dot{\epsilon}_{ij}\, dV = \int_D \left(\int_0^T \Delta\sigma_{ij}\, \Delta\dot{\epsilon}_{ij}\, dt\right) dV = \int_D W[x, \Delta\overset{T}{\underset{0}{\epsilon}}_{ij}]\, dV = 0. \tag{17}$$

If W is positive definite then (17) demands that

$$W[x, \Delta\overset{T}{\underset{0}{\epsilon}}_{ij}] = 0$$

in D and therefore $\Delta\dot{\epsilon}_{ij}(x, t)$ and hence $\Delta\epsilon_{ij}(x, t)$ and $\Delta\sigma_{ij}(x, t)$ must vanish identically in the time interval $[0, T]$ everywhere in D. The last conclusion implies that there cannot exist two distinct stress and strain fields satisfying (2), (3), (4) and (5) and (15).

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DISTORTION OF BOUSSINESQ FIELD BY CIRCULAR HOLE*

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Introduction. The classical Boussinesq solution to the problem of a concentrated load acting on the straight boundary of a semi-infinite plate is basic to a number of problems in the plane theory of elasticity. Barjansky [1] modified the Boussinesq problem and analyzed the effects of a circular hole in the plate. In the following paper the latter problem has been restated and some corrections affecting the results have been made.**

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**Calculations are shown in Appendix.