ON THE FOLDING OF A VISCOELASTIC MEDIUM WITH ADHERING LAYER UNDER COMPRESSIVE INITIAL STRESS*

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Abstract. The exact solution is given for the folding by compression of a viscoelastic layer embedded in a viscoelastic medium, under the assumption that there is perfect adherence between layer and medium. This solution agrees closely with the earlier result obtained by Biot which was based on the assumption that layer and medium could slip over each other.

In a previous publication, Biot [1] has discussed the case of folding a layered viscoelastic medium under initial stress. An exact solution was presented for the folding, due to instability, of a viscoelastic layer of thickness h embedded in a viscoelastic medium extending infinitely far in both directions perpendicular to the layer; the assumption was made that layer and medium did not adhere and could slip over one another as if perfectly lubricated. An approximate solution for the case of perfect adherence was developed in a second publication of the same year by Biot [2]. The influence of the adherence was found to be small. The purpose of this note is to check this conclusion by an exact solution for the case when perfect adherence exists between layer and medium.

The method is the same as that in Ref. [1] and it combines two distinct developments contained in earlier work by the same author. One of these developments is the theory of elasticity of a medium under initial stress (1934-1941). The other was introduced as a correspondence principle (1954-56) by which elastic moduli are replaced by corresponding operators [3, 4, 5]. Results are therefore applicable to either elastic or viscoelastic media of a very general nature. It was shown at the same time that the correspondence extends to problems which involve anisotropy, dynamics, wave propagation, and variational procedures. The mathematical restrictions on the operators were also derived from thermodynamics [3].

The horizontal and vertical displacements u and v, in the layer representing the departure of the displacement from an initial steady state, satisfy the equations

$$\nabla^2\left[\left(\bar{Q} - \frac{P}{2}\right)\frac{\partial^2}{\partial x^2} + \left(\bar{Q} + \frac{P}{2}\right)\frac{\partial^2}{\partial y^2}\right](u, v) = 0, \tag{1}$$

which are a consequence of Eq. (3.7) of Ref. [1].

$\bar{Q}$ is a time operator, defined in [1], which reduces in case of Newtonian viscosity to $\mu\, d/dt$, where μ is the viscosity coefficient. P is the initial compressive stress. An equation similar to Eqs. (1) holds for the embedding medium, with the difference that now indexed symbols $\bar{Q}_1$ and P_1 are used.

The forces per unit initial area exerted on the medium by the layer are F_x and F_y (Fig. 1); those per unit initial area exerted on the layer are F'_x and F'_y. Equilibrium requires that

$$F_x = -F'_x$$

and

$$F_y = -F'_y \tag{2}$$

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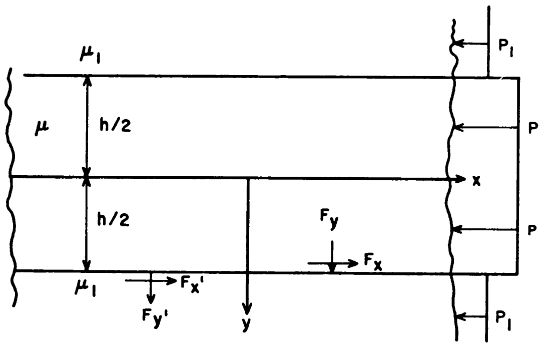

FIG. 1. Diagram of the forces acting between layer and embedding medium.

Solutions of Eqs. (1), which are pertinent, are

$$\begin{aligned} u &= \sum A_i e^{K_i l y} \sin lx \\ v &= \sum B_i e^{K_i l y} \cos lx, \end{aligned} \tag{3}$$

where

$$K_i = \pm 1, \pm\left(\frac{2\bar{Q} - P}{2\bar{Q} + P}\right)^{1/2}.$$

Because these solutions must also satisfy Eqs. (3.7) of Ref. [1], there are only four independent constants.

For the infinite medium only negative exponents apply, whereas for the layer, hyperbolic sines and cosines must be taken. By expressing the boundary conditions in terms of the strain components, we find that

$$\begin{aligned} F_x &= \tau_0 \sin lx & F_x' &= \tau_0' \sin lx \\ F_y &= q_0 \cos lx & F_y' &= q_0' \cos lx \end{aligned} \tag{4}$$

in which τ_0, q_0, τ_0', and q_0' are given in terms of P, $\bar{Q}$, P_1, $\bar{Q}_1$, and the integration constants of Eqs. (3). Because of symmetry, it is sufficient to consider only the conditions on one of the interfaces. At this interface, we write the layer displacement as

$$\begin{aligned} u &= U_0 \sin lx \\ v &= V_0 \cos lx. \end{aligned} \tag{5}$$

Thus, by Eqs. (3), the integration constants can be expressed in terms of U_0 and V_0. Introducing these in Eqs. (4), the resulting equations are

$$\tau_0 = C_{11} U_0 + C_{12} V_0$$

and

$$q_0 = C_{21} U_0 + C_{22} V_0, \tag{6}$$

where C_{ij} is a function of $\bar{Q}$, $\bar{R}$, and P. $\bar{R}$ is an operator related to the compressibility of the layer.

Because of the assumption of perfect adherence, the same equations (5) holds for the medium at the interface. Hence, an equation similar to (6) can be written for τ_0' and q_0'.

The coefficients C_{ij} in Eqs. (6) are not necessarily symmetric. The reason for this is that we have used a stress-strain relation involving a symmetric matrix, whereas the actual incremental stresses and strains as a consequence of thermodynamic principles cannot, in general, be expressed by means of symmetric elastic moduli or operators [2]. This difficulty disappears however, if we are dealing with an isotropic incompressible material. In this case, the matrix in Eqs. (6) is symmetric.

The two sets of equations of the type of Eq. (6)—one for the layer and one for the medium—reduce by Eqs. (2) to two linear homogeneous equations in the unknowns U_0 and V_0. The condition that these equations are compatible then leads to the stability equation. For the particular case where both layer and medium are perfectly incompressible and the prestress in the medium is assumed to be zero—that is, the compression is wholly supported by the stiff layer—the stability equation becomes

$$(1 - n^2) \tanh \gamma = [(1 + \zeta)^2 - n^2]k \tanh k\gamma + n\zeta(1 + k \tanh \gamma \cdot \tanh k\gamma), \tag{7}$$

in which

$$n = \frac{\bar{Q}_1}{\bar{Q}}, \quad \gamma = l\frac{h}{2}, \quad \zeta = \frac{P}{2\bar{Q}}, \quad \text{and} \quad k = \left(\frac{1 - \zeta}{1 + \zeta}\right)^{1/2}.$$

A diagram of Eq. (7) is shown in Fig. 2 for the case of purely viscous solids, i.e., for $n = \mu_1/\mu/$ (ratio of viscosities).

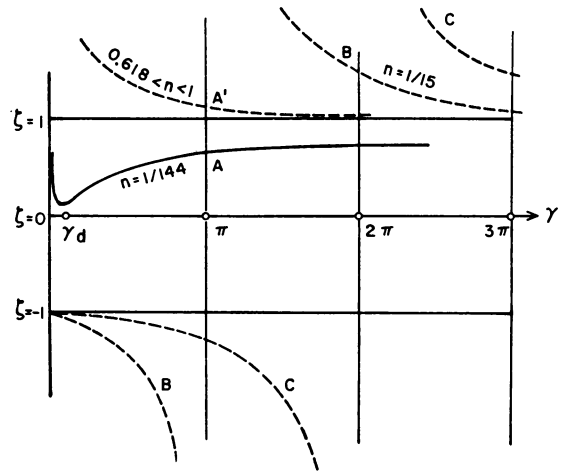

FIG. 2. Graph of Eq. (7) for purely viscous solids.

An infinite number of branches result. The only branch that is significant is curve A of Fig. 2, which is shown here for the particular value $n = 1/144$ of the viscosity ratio. It shows a dominant wavelength corresponding to γ_d, where ζ reaches a minimum value. Some of the other branches are also shown in the diagram—curves B and C for $n = 1/15$. They have the vertical asymptotes $\gamma = \pi, 2\pi$, etc. For a Newtonian fluid, they are physically spurious in the region $\zeta < -1$ or $\zeta > 1$, because where $|\zeta|$ is close to or larger than unity, the solutions lose their physical significance. On the other hand, for *a strongly non-linear solid which exhibits plastic flow*, the solutions may be physically significant in a larger range of ζ. But the question arises whether the incremental properties remain isotropic. These remarks should also be kept in mind in connection with the physical significance of some of the results of Ref. 1 in their application to Newtonian fluids.

The solution we have obtained is also directly applicable to the stability of purely elastic media when the operators are replaced by elastic moduli. In this case, for a material which is isotropic in the unstressed state, the only accessible values of $|\zeta|$ are smaller than unity. These points will be discussed more extensively in a forthcoming publication.

For values $0.618 < n < 1$, branch A moves into the region $\zeta > 1$ (as shown by curve A') and has no minimum. Hence, for the reason stated previously, the branch A' becomes physically spurious.

For materials of more general viscoelastic properties, a set of curves A, each computed for a different viscosity ratio n, can serve as a master plot in determining the dominant wavelength at a given strain rate.

Comparison of the results of the solutions for perfect slip and perfect adherence shows very little difference. Most of the difference is in the region of dominant wavelength, but even here the value of ζ varies by less than 2 percent, and the shift in dominant wavelength is of the same order of magnitude. For practical purposes, it therefore seems sufficient to use the theory without adherence; this agrees with the conclusions of Ref. [2]. It is interesting that outside the region of dominant wavelength, the solutions (for incompressible media) with or without slip become indistinguishable to a high order of accuracy.

The influence of compressibility under conditions probably prevailing in the earth's crust is negligible. The general form of Eq. (7), as we take into account the compressibility of layer and medium, contains a number of additional terms, each containing one of the factors

$$\alpha = \frac{\bar{Q}}{\bar{Q} + \bar{R}}, \qquad \alpha' = \frac{\bar{Q}_1}{\bar{Q}_1 + \bar{R}_1}, \tag{8}$$

or both. For elastic compressibility [3],

$$\bar{R} = K - \tfrac{2}{3}\bar{Q}, \tag{9}$$

where K is the bulk modulus of the material, which is here a constant. We have

$$\alpha = \frac{3}{1 + 6\dfrac{K}{P}\zeta}, \tag{10}$$

which shows that the effect of compressibility depends on the magnitude of the compressive load. For loads P much smaller than K, α and α' will be small quantities; then compressibility will not be important. For large values of P, however, the changes in dominant wavelength and strain rate may become appreciable.

References

1. M. A. Biot, *Folding of a layered viscoelastic medium derived from an exact stability theory of a continuum under initial stress*, Quart. Appl. Math., **17**, No. 2, 185–204 (1959)
2. M. A. Biot, *On the instability and folding deformation of a layered viscoelastic medium in compression*, J. Appl. Mech., Am. Soc. Mech. Engrs. **26**, 393–400 (1959b)

3. M. A. Biot, *Theory of stress-strain relations in anisotropic viscoelasticity and relaxation phenomena,* J. Appl. Phys. **25**, No. 11, 1385–1391 (1954)
4. M. A. Biot, *Dynamics of viscoelastic anisotropic media,* Proc. Second Midwestern Conf. Solid Mech., Research Series No. 129, Engineering Experiment Station, Purdue University, Lafayette, Ind., pp. 94–108, 1955
5. M. A. Biot, *Variational and Lagrangian methods in viscoelasticity,* in Deformation and Flow of Solids (R. Grammel, ed.), Springer-Verlag, Berlin, 1956, pp. 251–263

ON UNIQUENESS IN LINEAR VISCOELASTICITY*

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Summary. It is shown that solutions of a class of boundary value problems in linear vicoelasticity are unique, if the relaxation moduli in shear and compression are steadily decreasing functions of time which are convex from below and tend to non-negative constant asymptotic values.

1. Introduction. Consider isothermal deformations of a linear isotropic viscoelastic solid. Let $\sigma_{ij}(x, t)$ and $\epsilon_{ij}(x, t)$ denote the components of the stress and infinitesimal strain tensors respectively in the rectangular cartesian coordinates x_i . Here, as in the sequel, the single argument x stands for the triplet of coordinates (x_1 , x_2 , x_3), while t denotes the time. With a view of stating constitutive laws governing the mechanical behavior in a convenient form we introduce the deviatoric components of stress and strain

$$s_{ij} = \sigma_{ij} - \tfrac{1}{3}\,\delta_{ij}\sigma_{kk}\,, \qquad e_{ij} = \epsilon_{ij} - \tfrac{1}{3}\,\delta_{ij}\epsilon_{kk}\,, \tag{1}$$

where δ_{ij} denotes the Kronecker-delta. We shall be concerned with the following integral representation of the mechanical behavior [1]**

$$\begin{aligned} s_{ij}(x, t) &= \int_0^t G_1(t - \tau)\,\frac{\partial}{\partial \tau}\,e_{ij}(x, \tau)\,d\tau, \\ \sigma_{kk}(x, t) &= \int_0^t G_2(t - \tau)\,\frac{\partial}{\partial \tau}\,\epsilon_{kk}(x, \tau)\,d\tau, \end{aligned} \tag{2}$$

where $G_1(t)$ and $G_2(t)$ are the relaxation moduli in pure shear and isotropic compression, respectively. This representation contains the tacit assumption that the solid is in the unstressed and unstrained virgin state for $t < 0$. Note also that G_1 and G_2 need only be defined for non-negative values of their arguments.

In view of the relative scarcity and incompleteness of experimental information concerning the moduli $G_1(t)$ and $G_2(t)$ it is important to know what restrictions can be imposed upon G_1 and G_2 on physical and mathematical grounds.

One set of such restrictions arises from the considerations of uniqueness in boundary value problems of the quasi-static linear theory of viscoelasticity. The complete system of field equations for such a boundary value problem consists of the equations of equilibrium

$$\frac{\partial}{\partial x_j}\,\sigma_{ij}(x, t) = 0, \tag{3}$$

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**Numbers in square brackets refer to the Bibliography at the end of the paper.