QUARTERLY

OF

APPLIED MATHEMATICS

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By J. ARTHUR GREENWOOD and H. O. HARTLEY

This *Guide* catalogues a large selection of tables belonging to the field of mathematical statistics, and a small selection of mathematical tables lying outside statistics but often used together with statistical tables. The bulk of the tables treated were published between 1900 and 1954; occasional entries relate to works as early as 1799 and as late as 1960. As well as filling an important need for those actively engaged in the computational side of mathematical statistics, this work offers valuable reference to the professional computer faced with a statistical problem, and the statistician called upon to compute. *Published for the National Council and the National Academy of Sciences.* *1076 pages. $8.50*

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BOOK REVIEWS

Sequential decoding. By Johm M. Wozencraft and Barney Reiffen. The Technology Press of The Massachusetts Institute of Technology, Cambridge, and John Wiley & Sons, Inc., New York, London, 1961. v + 74 pp. $3.75.

One of a series of Technology Press Research Monographs, this book is a compendium of some significant research done over the past few years at MIT. It is based on the doctoral research of Wozencraft and the subsequent doctoral research of Reiffen, and also includes results obtained by others, notably M. Horstein and R. G. Gallagher.

The book represents a self-contained presentation of the concept of sequential decoding for reliable communication. Its basic aim is to provide means for decoding sequentially in such a fashion that the computational complexity grows only algebraically with n, the number of code digits in each of 2^s code words, assuming that the code is binary (for example) and that the rate $R = k/n$ is fixed. Thi aim is accomplished one information digit at a time: each information digit results from the elimination of half the possible messages, upon receipt of n successive code digits. Just enough of these digits are considered to guarantee that the particular information digit can be decoded correctly, given some desired probabilistic or other criterion of success.

The book is very interesting and well organized. It includes a thorough presentation of introductory fundamentals, a detailed discussion of sequential decoding for the binary symmetric channel, a brief treatment of the corresponding encoding, some experimental results, and extensions to more general channels. Although the extensions and related results are indicated only sketchily, references are given to more detailed expositions. All in all, the authors have made a substantial contribution by circumventing the problem of exponential growth of computational complexity. The short length of the book (which is in itself praiseworthy) is inversely proportional to the content, but makes careful reading essential.

PETER G. NEUMANN

Lectures on linear algebra. By I. M. Gel'fand. Interscience Publishers, Inc., New York, London, 1961. ix + 185 pp. $6.00.

These lectures on linear algebra begin with a discussion of the properties of n-dimensional spaces. Then the various classes of linear transformations on an n-dimensional vector space are demonstrated. After a chapter on arbitrary transformations of canonical form some of the previously described concepts are generalized as an introduction to tensors.

The presentation of the material by a rigorous scheme of definition, theorem and proof is further enhanced by frequent examples and exercises.

HEINZ WALDBURGER

Tables of higher functions. By E. Jahnke and F. Emde. Sixth edition. Revised by F. Lösch. B. G. Teubner Verlag., Stuttgart, and McGraw-Hill Book Co., Inc., New York, Toronto, London, 1960. xii + 318 pp. $14.00.

In this book F. Lösch revises Jahnke and Emde's collection of tables of higher functions. Most of the material has been rearranged, the functions and notations are explained at somewhat greater length and new accompanying difference tables permit easier interpolation. Also added are tables related to the following: Error Function; Fresnel Integrals; Theta Functions; Orthogonal Polynomials (Laguerre, Hermite and Chebyshev polynomials); Struve Function; Confluent Hypergeometric Functions; Special Functions (Plank Radiation Function, Langevin Function, Plank-Einstein Function, Debye Function, Source Functions of Heat Conduction).

HEINZ WALDBURGER

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BOOK REVIEWS

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Methods of feasible directions. By G. Zoutendijk. D. Van Nostrand Co., Inc., New York, New Jersey, and Elsevier Publishing Co., Amsterdam, London, New York, Princeton, 1960. 126 pp. $6.25.

This monograph is a well written and extremely complete exposition of various aspects of non-linear programming. The emphasis is on computational methods, and several original contributions of significance are included. It is to be highly recommended to mathematicians interested in such problems. Non-mathematicians will find a useful set of rules and procedures for mechanically solving both linear and non-linear programs.

Unfortunately, the publishers have chosen to set the print so close together as to make the book extremely difficult to read. This also accounts for the fact that such a wealth of material has been crowded into 126 pages.

Several new and important contributions to the field are made in the book. The first of these is the revised product form algorithm for the simplex method of linear programming. This is a modification of the product form algorithm due to Dantzig and Orchard-Hays which considerably reduces the computation required in the pricing operation.

The second contribution is the development of a series of methods for obtaining approximate solutions to non-linear programs. These are large step gradient methods and have been called "methods of feasible directions." A portion of this development was published previously (Jour. Royal Stat. Soc., *B, 21,* 1959).

The early sections of the book contain a brief review of the theory of non-linear programming including proofs of Farkas' lemma, the "key" theorem and the Kuhn-Tucker saddle-point theorem.

The remainder deals with computational methods and begins with a discussion of the simplex, dual and primal-dual methods for linear programming. Various algorithms for the simplex method are described and compared on the basis of number of multiplications, amount of bookkeeping and accuracy. Recommendations are made on which algorithm to use, based on the size and scope of the computer available.

Several methods of feasible directions are developed in great detail for the solution of non-linear problems. Convergence proofs are given for these iterative methods, and they are shown to be finite for quadratic programming problems. Many of the well-known methods for linear, quadratic and convex programming are shown to be included in these methods of feasible directions.

W. S. Dorn

Analogue computers. By I. I. Eterman. Translated from the Russian by G. Segal, and translation edited by B. H. Venning. Pergamon Press Inc., New York, Oxford, London, Paris, 1960. ix + 264 pp. $8.50.

This text deals primarily with problems concerning the use of analogue computers but it also introduces the general material about design and functioning that must be understood for the proper use of these machines. After a discussion of the properties of dynamic systems the circuits and elements of analogue computers are introduced. Generators for functions of many variables, for oscillations and for special non-linearities are demonstrated. Short descriptions of Russian electronic analogue computers are given and the transformation of the physical system into the corresponding machine system is illustrated by examples. Other material is devoted to the solution of linear systems, algebraic equations and boundary value problems of ordinary and partial differential equations—all by means of analogue computers. The estimation of errors in solutions of dynamic systems is investigated and appropriate methods are given.

Heinz Waldburger

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BOOK REVIEWS

(*Continued from page 348*)

Partial differential equations and continuum mechanics. Edited by R. E. Langer. The University of Wisconsin Press, Madison, Wisc. xv + 397 pp. $5.00.

The volume contains the full texts of invited lectures and abstracts of contributed papers presented at an international conference held at the Mathematics Research Center of the University of Wisconsin. The following subjects are treated in the lectures: differential equations in mathematical physics (C. Müller), angular distribution of eigenvalues of non self-adjoint elliptic boundary value problems of higher order (S. Agmon), indefinite differential eigenvalue problems: asymptotic distribution of their eigenfunctions (A. Pleijel), bounds for eigenvalues and method of intermediate problems (A. Weinstein), linear elliptic equations of higher order in two independent variables and singular integral equations with applications in elasticity (G. Fichera), propagation of surface waves in anisotropic media (R. Stoneley), finite deformation of plates into shells (B. R. Seth), statistical fluid mechanics: two-dimensional linear gravity waves (J. Kampé de Fériet), continuations of Laplace's transformation: their application to differential equations (J. Leray), regularity problem for elliptic and parabolic differential equations (J. Moser), atypical partial differential equations (H. Lewy), asymptotic behavior of flow past a body of a compressible viscous or electrically conducting fluid (I. Imai), transonic gas flow and equations of mixed type (F. G. Tricomi), transonic nozzle flows found by the hodograph method (T. M. Cherry), existence of solutions of partial differential equations (L. Hörmander), existence and differentiability theorems for variational problems for multiple integrals (C. B. Morrey, Jr.), contribution to mathematical methods applied in fluid mechanics (D. P. Riabouchinski), a functional equation related to the Boltzmann equation and the equations of gas dynamics (J. M. Burgers), parabolic equations with applications to boundary layer theory (K. Nickel).

The solution of equations in integers. By A. O. Gelfond. Translated from the Russian by J. B. Roberts. W. H. Freeman & Co., San Franscisco, London, 1961. viii + 63 pp. §1.00.

Based on lectures given at the Moscow State University in 1951, the booklet presents the more elementary parts of the field in a stimulating and eminently readable manner, that should be within the reach of high school students.

W. Prager

Progress in operations research. Volume I. Edited by R. L. Ackoff. (Publications in Operations Research, Operations Research Society of American, No. 5.) John Wiley & Sons, Inc., New York, 1961. xii + 505 pp. $11.50.

This extremely useful survey contains the following articles: *The Meaning, Scope, and Methods of Operations Research*, by R. L. Ackoff; *Decision and Value Theory*, by C. W. Churchman. *A Survey of Inventory Theory from the Operations Research Viewpoint*, by F. Hanssmann; *Mathematical Programming*, by E. L. Arnoff and S. S. Sengupta; *Dynamic Programming*, by S. Dreyfus; *Dynamics of Operational Systems: Markov and Queuing Processes*, by P. M. Morse; *Sequencing Theory*, by R. L. Sisson; *Replacement Theory*, by B. V. Dean; *The Theory and Application of Simulation in Operations Research*, by G. W. Morgenthaler; *Military Gaming*, by C. Thomas; *Progress in Operations Research: The Challenge of the Future*, by J. F. Magee and M. L. Ernst. The editor is to be congratulated for his choice of authors and topics, and thanks are due to ORSA and the publishers for taking the initiative in starting this series of surveys in a rapidly growing field.

W. Prager

An introduction to linear statistical models. By Franklin A. Graybill. Volume I. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1961. xiii + 463 pp. $12.50.

Exceedingly complete coverage of the topics presented is a primary feature of *An Introduction to Linear Statistical Models, Volume I.* The only exception to this completeness is perhaps the chapter on computing techniques which may be excused considering the present availability of high speed computing devices.

The mathematical development of statistical theory in this book relies heavily on the use of matrices, vectors and determinants. A condensed presentation of theorems, generally without proof, is given in the first chapter with special emphasis on derivatives of matrices and vectors, idempotent matrices and the use of Jacobians.

Since quadratic forms are of fundamental importance in statistical theory, space is devoted to their properties, distributions and relation to certain linear forms. Contrary to the usual presentation, the non-central chi-square, non-central F and Wishart's distributions are introduced, particularly with reference to the power of significance tests along with tables to facilitate such computations.

Chapters 5 through 18 are concerned with the mathematical treatment of the linear model, generally in the form of point and interval estimation, hypothesis testing, etc. Five classes of models are treated from the point of view of infinite-model theory. Model 1 relates an observable random variable, y, to the mathematical variables $X_1, X_2, \cdots, X^k$. The treatment of this case includes most of the usual uses of the general linear hypothesis including polynomials models. Model 2 is the case where errors of measurement are present in both independent and dependent variables. Two situations are considered: (1) the ratio of the measurement error variances known; (2) the controlled-independent-variable model. When the dependent and independent variables are a set of jointly distributed random variables, Dr. Graybill refers to the model as a Regression Model or Model 3. Model 3 differs from Model 1 only in that the X's are particular values of random variables rather than mathematical variables. This definition of regression is somewhat different than that usually given where no assumptions are made regarding the distribution of the X's and in fact they may be non-stochastic. Models 4 and 5 are the usual experimental design and variance-component models and over half the book is devoted to them. The crossed-classification model, nested model, models with and without interaction, unequal subclass numbers, test for additivity, fixed, random and mixed models are a sample of topics covered.

Dr. Graybill has produced an excellent text of especial use as a reference. The use of matrix theory has made much of the presentation compact and elegant but may make it difficult for students and experimenters to acquire an understanding of the applications of statistics. The reviewer is looking forward to the appearance of Volume II.

A. H. E. GRANDAGE

Introduction mathématique a la mécanique des fluides. By Caius Jacob. Gauthier-Villars, Paris, 1959. 1286 pp. $10.00.

This work was originally published in 1952 in the Roumanian language. The present translation into French will make it available to a much wider public. Moreover the translation includes some important new material namely additional considerations on mixed boundary value problems concerning the Dirichlet problem with given singularities; flows with constant vorticity and a new section on transonic flows.

The book consists of five parts: 1) on some boundary value problems; 2) equations of motion; 3) theories of hydrodynamic resistance for incompressible fluids; 4) compressible fluids; 5) methods of approximation in the dynamics of compressible fluids.

The orientation is definitely aerodynamical. Wave motion, for example, is not mentioned.

The author was a pupil of Villat at the Sorbonne and this is a fortunate circumstance, for he attended in 1932 Villat's lectures on Chaplygin's work, thus not only obtaining a knowledge of its principles but also profiting by Villat's masterly amplifications. Chaplygin published his work in 1902–1904, but in Russian, and it remained almost completely unknown until the work of Riabouchinsky and Demtchenko

brought it to light. The result, parts 4 and 5, is here and now a really thorough and illuminating account of Chaplygin's researches and their applications.

The book can be recommended for the rigour of the mathematics and for detailed accounts of some of the major methods and applications, all well documented. It is a pity that there are no exercises, and the reviewer feels that the lack of an index in a work of 1286 pages is real handicap.

L. M. MILNE-THOMSON

Hypersonic flow. Edited by A. R. Collar and J. Tinkler. Proceedings of the Eleventh Symposium of the Colston Research Society held in the University of Bristol in April 1959. Academic Press Inc., New York, and Butterworths Scientific Publications, London, 1960. xv + 432 pp. $13.50.

This book represents the Proceedings of the Eleventh Annual Symposium of the Colston Research Society held in the University of Bristol in April 1959. It consists of fifteen papers, and discussions of them, covering various aspects of hypersonic flow studies including theory, experiment, and design. Although the majority of the papers are devoted to British research in the field, contributions are also to be found from both the United States and France. Most of the experimental papers deal with the development of various hypersonic facilities such as shock tunnels, gun tunnels, ballistic ranges, etc., as well as with methods of measurement. These papers along with those which consider the more practical design problems of long-range hypersonic flight are all of high quality.

Of particular interest and recommended to the interested applied mathematician is the theoretical paper of Miles which discusses unsteady hypersonic flows; the contributions of Mangler and Van Dyke on the calculation of inviscid blunt body flows by inverse methods; and Guiraud's paper on three-dimensional Newtonian flows.

The discussions following the papers are very well presented and contribute significantly to clarifying various aspects of the different papers. The editors are to be congratulated on their excellent organization of these discussions. This reviewer would definitely recommend this collection to those with an interest in the field who wish to acquaint themselves with some of the more recent researches, particularly those in England.

RONALD F. PROBSTEIN

Applications of finite groups. By J. S. Lomont. Academic Press, Inc., New York, London, 1959. xi + 346 pp. $11.00.

The book assumes a familiarity with matrix algebra and group theory but includes introductory chapters on these topics in which the basic concepts and theorems are stated briefly, and outline proofs of some of the less obvious theorems are given. Group representation theory and the construction of character tables are treated in some detail and this provides the mathematical basis for the applications discussed later. These are mainly concerned with wave mechanics and crystallography, and a knowledge of quantum mechanics is presupposed. A derivation is given of the crystallographic point and space groups, and molecular vibrations; molecular orbitals and wave functions in crystals are also treated. The theory of symmetric groups is developed briefly and applied to examine atomic, molecular and nuclear structures. Some more elementary topics are also included, such as transformations of thermodynamic variables and reflection and transmission coefficients for symmetric waveguide junctions.

The book is concerned mainly with the finite groups and their applications, but the three dimensional rotation group is also considered and an appendix has been added on the Lorentz groups.

J. E. ADKINS