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BOOK REVIEWS

The algebra of probable inference. By Richard T. Cox. The Johns Hopkins Press, Homewood, Baltimore 18, Maryland, 1961. x + 114 pp. $5.00.

This work is a monograph for the philosopher-logician interested in studying the conceptual aspects of the fundamentals of the mathematical theory of probability. In the first of the three parts, the use of logical symbols in the elementary operations of probability is discussed; in the second entropy is introduced and analyzed, and in the concluding part, the author treats expectations. The whole reads quite pleasantly and is enlivened by a sprinkling of literary quotations.

RICHARD BELLMAN

Relativity: the general theory. By J. L. Synge. North-Holland Publishing Co., Amsterdam, and Interscience Publishers Inc., New York, 1960. xv + 505 pp. $16.50.

The years following Einstein's publication of his general relativity theory have yielded a number of texts devoted to expositions of its intricacies and simplicities: some excellent, some of a tolerable nature, and some to be cast into justly deserved oblivion. Professor Synge's book is definitely of the first category, presenting in his characteristically lucid style many of the more subtle points of this difficult subject. The principal theme is the examination of those problems which are associated with a classic theory of measurement in an intrinsically curved manifold not admitting absolute simultaneity. In accomplishing this arduous task, a two-point invariant world-function (essentially the square of the geodesic separation of two events) is introduced and systematically used, thereby permitting power-series expansions and approximation analyses without abandoning the tensor calculus. With this tool and a liberal use of space-time diagrams, the contribution made by this book fills a definite gap in the present literature—both from the standpoint of laying a plausible foundation for a classic theory of measurement and from the standpoint of clarifying many points of a local nature relative to the interpretation of the Einstein theory.

The presentation is confined to the classic aspects of relativity theory and constitutes a natural continuation of the author's previous text, *Relativity: The Special Theory* (for review, see Quarterly of Applied Mathematics, Vol. XV, No. 1, p. 40 (1957)). Great emphasis is placed on problems associated with local questions, geodesic geometry, and the trajectories of test particles and photons. The results obtained along these lines are significant, in many instances new, and a basic requirement for any serious student of relativity theory. Despite its 505 pages, however, many of the global and field-theoretic questions are either not mentioned at all or only touched upon in passing. Unless one is careful and thoroughly familiar with the fundamental global and field-theoretic problems of the theory, it is quite easy to lose the over-all aspects of the Einstein contribution and to reach the errongeus conclusion that one has spanned the content and generic ideas of general relativity by considerino geodesics and systems of vectors defined along curves in a four-dimensional metric space with ± 2 signature. For this reason alone, not to mention the fact that the basic assumptions of the theory are not spelled out in any detail, the book is not a text for the uninitiated. To quote from the preface, "There are heavy calculations in the book, but there will be places where the reader will find me sitting on the fence, whistling, instead of rushing into the fray." The treatment of gravitational waves and, in particular, the treatment of electromagnetic phenomena are most aptly described by this statement. It is the reviewer's opinion that it would have been better to eliminate the "fence sitting" sections since they make no definite contribution to the theory in their present form, and to some extent are inconsistent in that equally important questions such as the invariant formulation of boundary conditions and the elimination of spurious solutions are not mentioned at all. The book would not lose by such deletion, and would present a shorter and more unified exposition of the basic ideas so forcefully presented in the remainder.

The following list of chapter headings will show the scope of this work: I. Essential tensor formula for Riemannian space-time. II. The world function. III. Chronometry in Riemannian space-time. IV. The material continuum. V. Some properties of Einstein fields. VI. Integral conservation laws and equations of motion. VII. Fields with spherical symmetry. VIII. Some special universes. IX. Gravita-

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tional waves. X. Electromagnetism. XI. Geometrical optics. These are followed by an excellent bibliography and index, the bibliography laudably presenting not only the immediate references to the literature, but also including references to reviews of the cited papers and books.

D. G. B. EDELEN

Méthodes de cal cul numérique—I. Algèbre non linéaire. By A. Korganoff. With the collaboration of L. Bosset, J.-L. Groboillot, and J. Johnson. Dunod, Paris, 1961, xxviii + 375 11. $11.86.

As is proper but by no means customary in this age of automatic computation, this volume begins with a chapter (21 pp.) on electronic computers written by L. Bosset of the Bull Company. There follows a chapter (71 pp.) on the theory of errors written by the principal author in collaboration with J.-L. Groboillot of the Bull Company. The core of the volume consists of two chapters by the principal author on the solution of non-linear equations in general (82 pp.) and algebraic equations in particular (90 pp.). A final chapter by J. Johnson (72 pp.) concerns the determination of eigenvalues of matrices. An unusual feature for a work on numerical analysis is the complete absence of numerical examples in the main body of the text. An appendix of 38 pp. contains what seems to be a bare minimum of numerical illustrations of the principles discussed in the preceding chapters.

Throughout the volume, the presentation is clear and concise. It is to be hoped that the lack of a subject index will be remedied in a future edition of this useful work.

W. PRAGER

Modern mathematics for the engineer, Edited by E. F. Beckenbach. McGraw-Hill Book Co., Inc., New York, 1961. xviii + 456 pp. $9.50.

As the 1956 volume of the same title, the present volume corresponds to a series of invitation lectures at the University of California. Part I entitled Mathematical Methods contains the following chapters: From Delta Functions to Distributions (by A. Erdelyi), Operational Methods for Separable Differential Equations (by B. Friedman), Integral Transforms (by J. W. Miles), Semigroup Methods in the Theory of Partial Differential Equations (by R. S. Phillips). Part II is devoted to Statistical and Scheduling Studies and consists of chapters on Chance Processes and Fluctuations (by W. Feller), Information Theory (by D. Blackwell), The Mathematical Theory of Control Processes (by R. Bellman), Formulating and Solving Linear Programs (by G. B. Dantzig), and The Mathematical Theory of Inventory Processes (by S. Karlin). Part III has the title Physical Phenomena; it has the following chapters: Monte Carlo Calculations in Problems of Mathematical Physics (by S. M. Ulam), Difference Equations and Functional Equations in Transmission-line Theory (by R. Redheffer), Characteristic-value Problems in Hydrodynamic and Hydromagnetic Theory (by S. Chandrasekhar), Applications of the Theory of Partial Differential Equations to Problems of Fluid Mechanics (by P. R. Garabedian), The Numerical Solution of Elliptical and Parabolic Partial Differential Equations (by D. Young), Circle, Sphere, Symmetrization and Some Classical Physical Problems (by G. Polya).

W. PRAGER

Problèmes de théorie générale des oscillations et de chronométrie. By Jules Haag and Raymond Chaleat. Gauthier-Villars, Paris, 1960. 412 pp. $8.00.

The form of this attractive and interesting book is uncommon in the recent English-language literature: it consists of 120 carefully stated problems from the field of vibrations, each followed immediately by the detailed solution. None of the problems is of the type usually associated with text book exercises and several are of the caliber of papers found in the scientific literature. All problems are concerned with and stated in the language of physical applications; while the majority is concerned

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with problems arising from the theory of the clock (pendulums, balance wheels, escapements, etc.) many of these are directly applicable to other vibrating systems as well.

The first chapter contains theory and application of astronomic time measurements (gnomon, sun dial, calendar, etc.) Chapters 2–12 and Chapter 14 deal with physical problems whose governing equation is $\theta'' + \omega^2\theta = \lambda f(\theta, \theta', t)$ where f is usually not linear in θ, θ' and is always periodic in t; moreover, f is not in all problems everywhere continuous in θ' and t. The problems deal with steady-state, and many with transient solutions. It is evident that some are of considerable difficulty. Chapter 13 treats problems of several degrees of freedom, and Chapter 15 contains problems not easily fitted under the well-defined headings of the preceding chapters.

The combination of analytical skill and sound physical understanding makes this book a welcome addition to the growing literature in nonlinear vibrations.

The senior author died several years prior to the publication of this book. Nevertheless, his name appears on the title page because he was the teacher of Chaleat, and because he proposed a number of the problems treated in this book.

R. M. ROSENBERG

Mathematical methods for engineers and technologists. By P. I. Romanovskii. Pergamon Press, New York, London, Paris, 1961. xiv + 253 pp. $8.50.

This excellent book differs from the many recent textbooks of similar title mainly in the rigour of its presentation. It shows that attention to detail and to finer points in mathematical analysis need not make the discussion more difficult. This is particularly apparent, for instance in the chapter on the Laplace transform which is a model of clarity. The other chapters are on Fourier series and integrals, vector analysis, the foundations of analytic function theory and some special functions. Although originally addressed to communications engineers, the book will be of equal value to other applied mathematicians.

W. FREIBERGER

Impact—the theory and physical behaviour of colliding solids. By Werner Goldsmith. St. Martin's Press, New York, 1960. xv + 379 pp. $17.50.

A proper understanding of impact phenomena requires a knowledge of a number of different fields of mechanics, and hitherto these had not been collected in a single volume. Professor Goldsmith's monograph is therefore extremely welcome, especially since it considers both elastic and anelastic impact.

After a short historical introduction in the first chapter and a general discussion of the kinematics and dynamics of the problem in the second, a detailed treatment of the subject of elastic impact is given in the third and fourth chapters. These cover elastic wave propagation and Hertzian stress distributions respectively. The fifth chapter, which is by far the longest in the book, is devoted to plastic impacts, and includes a full discussion of the propagation of plastic waves. The last two chapters are concerned with experimental results and measurements of the dynamic properties of materials. Finally, the book contains an extensive bibliography of the whole field. The book touches on so many topics concerned with impact that many readers will inevitably feel that some subjects have been treated far too briefly; thus the present reviewer would have liked to see a fuller treatment of viscoelastic behavior. Individual tastes apart, the book can be recommended as a valuable introduction to the subject, as well as a reference book useful to both the engineer and the physicist who are concerned with impact and wave propagation problems.

H. KOLSKY

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Modern computing methods. (National Physical Laboratory, Teddington, Middlesex,—Notes on Applied Science, No. 16.) Second edition. Her Majesty's Stationery Office, London, 1961. vi + 170 pp., $3.78.

The first edition of this well-known work was written by L. Fox, E. T. Goodwin, J. G. L. Michel, F. W. J. Olver, and J. H. Wilkinson. The present edition has been prepared by C. W. Clenshaw, E. T. Goodwin, D. W. Martin, G. F. Miller, F. W. J. Olver, and J. H. Wilkinson. The contents of the earlier edition have been thoroughly revised to take account of the advances in automatic computation. The chapter on *Relaxation Methods* has been replaced by one entitled *Linear Equations and Matrices: Iterative Methods*, and the chapter on *Computation of Mathematical Functions* has been expanded into two chapters with the headings *Evaluation of Limits*; *Use of Recurrence Relations* and *Evaluation of Integrals*. New chapters entitled *Linear Equations and Matrices: Error Analysis* and *Chebyshev Series* have been added.

W. PRAGER

Complex variables and the Laplace transform for engineers. By W. R. Lepage. McGraw-Hill Book Co., Inc., New York, Toronto, London, 1961. xvii + 475 pp. $12.50.

On the subjects covered, this book tries to anticipate and answer the questions which might occur to a graduate electrical engineering student. Theorems are proved rigorously, but the emphasis is on interpretation and understanding rather than on generality. The fundamentals are discussed thoroughly and illustrated by examples worked out in detail. The reader is led from the elements of complex variables through conformal mapping, integration, infinite series, and multivalued functions at a gradually increasing level of sophistication. This material is followed by a chapter on the convergence of real integrals. The second part of the book starts with the Fourier integral and proceeds through the two-sided Laplace transform to the one-sided transform. There is a chapter on convolutions, and several chapters on subjects more directly oriented toward electrical engineering, such as the Z transform.

Some topics of interest to the applied mathematician are not covered in enough detail to be useful. For example, the chapter on conformal mapping is meant primarily to convey the geometric notion of analytic function as a mapping, and the use of conformal mapping in potential theory is only briefly mentioned. Nevertheless, the range of material is so wide and the presentation is good enough that this book should be valuable to any student of complex functions or the Laplace transformation.

ALLEN PIPKIN

Boundary and eigenvalue problems in mathematical physics. By Hans Sagan. John Wiley & Sons, Inc., New York, London, 1961. xviii + 381 pp. $9.50.

This book is based upon a course taught to seniors and first-year graduate students in mathematics and the physical sciences who are familiar with some advanced calculus including vectors. As compared with most books of mathematics for engineers or physicists this book covers a narrow range of topics but treats them quite thoroughly. The central theme throughout the book is the solution of the common types of partial differential equations that occur in physics (the heat equation, wave equation, etc.) by separation of variables and the corresponding variational problems. In pursuit of these solutions the author treats the calculus of variation, Fourier series, the Sturm-Liouville theory, Legendre and Bessel equations, extremal properties of eigenvalues, and spherical harmonics. The most frequent reference is to Courant and Hilbert "Methods of Mathematical Physics," the influence of which is very clear.

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Most of the book is clearly written, well motivated and mathematically precise. The treatment of the Sturm-Liouville theory and the extremal properties of its eigenvalues is especially good. The quality, however, deteriorates towards the end of the book, in particular, there is a poor description of the Schroedinger equation. The quality is certainly above the average book at this level, and the work should serve well as either a text or supplementary reference.

G. F. NEWELL

Analysis for production management. By E. H. Bowman and R. B. Fetter. Revised edition. R. D. Irwin, Inc., Homewood, Ill., 1961. xiii + 562 pp. $8.75.

This revised edition contains about 60 pages more than the first edition, which appeared in 1957. There are numerous minor insertions, which take account of new developments. Moreover, there are many instances in which the exposition has been improved, and the number of problems has been greatly increased.

W. PRAGER

Radio waves in the ionosphere. By K. G. Budden. Cambridge University Press, New York, 1961. xxiv + 542 pp. $18.50.

During the last decade a wide interest has arisen in plasma physics. This has developed in connection with both thermonuclear research and space research. Many new comers in the field are comparatively unaware of the long history of electromagnetic investigations into plasma phenomena that have been in active development in connection with the ionosphere for about four decades. A principle reason for this is that only a fraction of the scientific literature in the field has so far found its way into books. Dr. Budden's book is quite a comprehensive treatment of the mathematical research that has been done over the past 40 years in connection with the propagation of electromagnetic waves through plasmas.

The book lays particular emphasis on the propagation of electromagnetic waves through plasmas for which the electron density varies from place to place, usually with plane stratification. The book deals with propagation through a plasma both in the absence and in the presence of an imposed magnetic field, which is usually thought of as the earth's magnetic field. The approximate methods appropriate to slowly varying media are dealt with in detail. The reflection and coupling phenomena that occur when slowly varying theory is inapplicable are also dealt with in detail. Many exact wave solutions are incorporated for particular profile of electron density, together with a detailed discussion of the electromagnetic field near a caustic. The book constitutes the most comprehensive treatment so far published of the mathematical theory of wave propagation in an ionized medium. The book may be strongly recommended to all interested in plasma physics, particularly those who have entered the field within the last decade.

H. G. Booker

Nuclear reactor theory. Edited by G. Birkhoff and E. P. Wigner. Proceedings of the Eleventh Symposium in Applied Mathematics of the A. M. S., New York, 1959. American Mathematical Society, Providence, 1961. v + 339 pp. $8.70.

This volume consists of the papers presented at the Eleventh Symposium in Applied Mathematics of the A. M. S. The nineteen papers of the Symposium were delivered by invitation only. It was the purpose of the Symposium and is the intention of this volume to interest research mathematicians in the fascinating and important field of nuclear reactor theory. Although the papers of the two editors should interest research mathematicians, there are a number of the other papers which are written in a manner of more interest to nuclear physicists.

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The authors of the papers are all well known contributors in the field of nuclear reactor theory. In general, the contributions here have the character of good review articles rather than new results. This has the advantage of presenting a rather complete status report on nuclear reactor theory. Overlooked in the volume, however, are articles dealing with reactor engineering and burnout or depletion. Both of these areas lead to non-conventional as well as conventional mathematical problems.

The book begins with a discussion of, "Reactor Types," by A. M. Weinberg. Two of the complications in thermal reactors, neutron thermalization and resonance absorption, are presented respectively by M. S. Nelkin and L. W. Nordheim. The penetration of radiation through reactor shields is discussed in the paper by U. Fano and M. J. Berger. The largest number of papers deal with the basic steady-state transport equation and the approximations that are used in place of it. This equation, which represents the steady-state behavior of reactors, leads mathematically to inadequately understood characteristic value problems. The papers of E. P. Wigner, J. Ernest Wilkins, Jr., Garrett Birkoff, G. J. Habetler and M. A. Martino, G. Milton Wing, Richard Bellman and Robert Kalaba, consider the analytical theory, while the papers of R. Ehrlich, Richard S. Varga and Bengt Carlson consider the actual methods of numerically solving approximate equations. The short time behavior of reactors is discussed in the last five papers by Harry Soodak, H. L. Garabedian, Harvey Brooks, T. A. Welton and W. K. Ergen.

WARD C. SANGREN

Transmission of information: a statistical theory of communications. By Robert M. Fano. The Massachusetts Institute of Technology Press, Cambridge, and John Wiley & Sons, Inc., New York, London, 1961. viii + 389 pp. $7.50.

This book is the outgrowth of a series of lecture notes prepared for a graduate course offered by the Electrical Engineering Department at M. I. T. The author states that it is specifically directed to graduate students and engineers interested in electrical communications. In fact, however, no prior knowledge of electrical communications or of any form of engineering is required of the reader. The book is a lucid, carefully written account of a large segment of Information Theory that should serve well any student with a reasonable amount of mathematical maturity and a modest knowledge of probability theory (no measure theory required).

A list of the chapter headings describes the material covered: the Transmission of Information; A Measure of Information; Simple Message Ensembles; Discrete Stochastic Sources; Transmission Channels; Channel Encoding and Decoding; Encoding for Binary Symmetric Channels; Multinomial Distributions; Encoding for Discrete, Constant Channels. The last two chapters are of a more advanced nature and contain much important material not previously published. The final chapter investigates the asymptotic error probability associated with the general discrete, constant channel. Well chosen problems are provided for each of the chapters.

For this reviewer, Prof. Fano's book was a joy to read. He avoids the overly terse, rigid, parsimonious style of so many texts in pure mathematics; yet, neither is he verbose nor, with the exception noted below, is rigor sacrificed. Each new concept is carefully motivated and each chapter ends with a valuable *Summary* section that points out what has been accomplished, which results are the really important ones and why. This is pedagogy at its best. Only one discordant note disturbed this reviewer. This is the occasional use of the engineer's fiction of infinite-band white noise resulting in a few equations devoid of meaning except possibly in some unstated limiting sense. But this is a small complaint hopefully to be corrected in a later edition. An introductory text as well done as this one deserves a long life.

DAVID SLEPIAN