

## ON ASYMPTOTIC BEHAVIOR OF SOLUTIONS OF SECOND-ORDER DIFFERENTIAL EQUATIONS\*

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**1. Introduction.** Consider the second-order linear differential equation

$$u'' - (1 + f(t))u = 0, \quad (1.1)$$

where

$$\int_0^\infty |f(t)| dt < \infty. \quad (1.2)$$

Then, as is well known (see [1]), there are solutions of (1.1),  $u_1$ , and  $u_2$ , with the respective properties that

$$u_1 \sim e^t, \quad u_2 \sim e^{-t} \quad (1.3)$$

as  $t \rightarrow \infty$ . A problem of some difficulty is that of matching given initial conditions at  $t = 0$  with given terminal conditions at  $t = \infty$ .

In a previous paper [2], we showed how functional-equation techniques could be used to treat corresponding questions for nonlinear differential equations of the form

$$\frac{dx}{dt} = Ax + g(x), \quad x(0) = c. \quad (1.4)$$

In this paper we wish to use these versatile techniques to discuss the class of equations

$$u'' - \left(1 + \sum_{k=1}^R z_k e^{-\lambda_k t}\right)u = 0, \quad u(0) = c_1, \quad u'(0) = c_2, \quad (1.5)$$

where the  $z_k$  are complex variables and  $\text{Re}(\lambda_1) < \text{Re}(\lambda_2) < \dots < \text{Re}(\lambda_R) < 0$ . Writing

$$\lim_{t \rightarrow \infty} e^{\lambda_1 t} u(t) = c_1 f_1(z_1, z_2, \dots, z_R) + c_2 f_2(z_1, z_2, \dots, z_R), \quad (1.6)$$

we wish to study the functions  $f_1$  and  $f_2$  and to obtain equations which determine them.

When (1.5) is written in the form

$$\begin{aligned} u'' - u - \sum_{k=1}^R v_k u &= 0, & u(0) &= c_1, & u'(0) &= c_2, \\ v'_k + \lambda_k v_k &= 0, & v_k(0) &= z_k, \end{aligned} \quad (1.7)$$

the general techniques of [2] may be applied. It is worthwhile, however, to introduce the special methods used here since they can be applied in other cases where the approach of [2] does not seem applicable.

**2.  $u'' - (1 + ze^{-\lambda_1 t})u = 0$ .** To illustrate the method, it suffices to discuss the case where  $R = 1$ . Write

$$\lim_{t \rightarrow \infty} u(t)e^{-t} = c_1 f_1(z) + c_2 f_2(z), \quad (2.1)$$

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where  $u$  satisfies (1.5) with  $R = 1$ , and  $z_1$  is replaced by  $z$  for typographical convenience. The effect of replacing  $t$  by  $t + \Delta$ , where  $\Delta$  is an infinitesimal, is to convert the equation

$$u'' - (1 + ze^{-\lambda_1 t})u = 0 \tag{2.2}$$

into the equation

$$u'' - (1 + ze^{-\lambda_1 \Delta} e^{-\lambda_1 t})u = 0, \tag{2.3}$$

which is to say  $z \rightarrow ze^{-\lambda_1 \Delta}$ ; simultaneously, the initial conditions  $u(0) = c_1$ ,  $u'(0) = c_2$ , are transformed into

$$u(0) = c_1 + c_2 \Delta, \quad u'(0) = c_2 + \Delta(1 + z)c_1, \tag{2.4}$$

to terms in  $\Delta^2$ . Consequently, from the definition of  $f_1$  and  $f_2$ , we have the relations

$$\begin{aligned} c_1 f_1(z) + c_2 f_2(z) &= \lim_{t \rightarrow \infty} u(t) e^{-t} \\ &= \lim_{t \rightarrow \infty} u(t + \Delta) e^{-(t + \Delta)} \\ &= e^{-\Delta} \lim_{t \rightarrow \infty} u(t + \Delta) e^{-t} \\ &= e^{-\Delta} [(c_1 + c_2 \Delta) f_1(ze^{-\lambda_1 \Delta}) + (c_2 + \Delta(1 + z)c_1) f_2(ze^{-\lambda_1 \Delta})]. \end{aligned} \tag{2.5}$$

Since this last relation holds for all  $c_1$  and  $c_2$ , we must have

$$\begin{aligned} f_1(z) &= e^{-\Delta} f_1(ze^{-\lambda_1 \Delta}) + \Delta(1 + z) f_2(ze^{-\lambda_1 \Delta}) + o(\Delta), \\ f_2(z) &= \Delta f_1(ze^{-\lambda_1 \Delta}) + e^{-\Delta} f_2(ze^{-\lambda_1 \Delta}) + o(\Delta). \end{aligned} \tag{2.6}$$

When we expand

$$\begin{aligned} f_1(ze^{-\lambda_1 \Delta}) &= f_1(z - \lambda_1 z \Delta) = f_1(z) - \lambda_1 z \Delta f_1'(z) + o(\Delta), \\ f_2(ze^{-\lambda_1 \Delta}) &= f_2(z - \lambda_1 z \Delta) = f_2(z) - \lambda_1 z \Delta f_2'(z) + o(\Delta), \end{aligned} \tag{2.7}$$

the relations in (2.6) yield the ordinary differential equations

$$z f_1' = \frac{(1 + z) f_2 - f_1}{\lambda_1}, \quad z f_2' = \frac{f_1 - f_2}{\lambda_1}, \tag{2.8}$$

upon letting  $\Delta \rightarrow 0$ . The initial conditions are easily obtained. When  $z = 0$ , the exact solution of (2.1) is

$$u = \frac{(c_1 + c_2) e^t}{2} + \frac{(c_1 - c_2) e^{-t}}{2}. \tag{2.9}$$

Thus  $f_1(0) = f_2(0) = \frac{1}{2}$ .

Write

$$f_1(z) = \sum_{n=0}^{\infty} a_n z^n, \quad f_2(z) = \sum_{n=0}^{\infty} b_n z^n. \tag{2.10}$$

Then (2.8) yields, for  $n \geq 1$ , the recurrence relations

$$n a_n = \frac{b_n - a_n}{\lambda_1} + \frac{b_{n-1}}{\lambda_1}, \quad n b_n = \frac{a_n - b_n}{\lambda_1}. \tag{2.11}$$

The determinant is  $n(n + 2/\lambda_1)$ , which means that we can always determine the  $a_n$  and  $b_n$  recursively if  $\text{Re}(\lambda_1) > 0$ , and as we see, under more general conditions as well. Furthermore, it is easily verified that the series obtained for  $f_1(z)$  and  $f_2(z)$  are entire functions of  $z$ .

3.  $x' - (A + e^{-Bt}Z)x = 0$ . It is clear that the same method can be applied to the general vector equation

$$x' - (A + e^{-Bt}Z)x = 0, \tag{3.1}$$

where  $A, B$ , and  $Z$  are constant matrices. Furthermore, equations of the form

$$u'' - (1 + ze^{-t^2+wt})u = 0 \tag{3.2}$$

can be treated at the expense of introducing partial differential equations.

4. **Approximation Methods.** Returning to the general equation of (1.1), suppose that  $f(t)$  approaches zero as  $t \rightarrow \infty$  rapidly enough so that we can closely approximate to it by means of a linear combination of exponentials

$$f(t) \sim \sum_{k=1}^R z_k e^{-\lambda_k t}. \tag{4.1}$$

Then we can use this approximation, and the power series developments, or numerical solution of the differential equations corresponding to (2.11), to obtain an approximation to the functional defined by the relation

$$\lim_{t \rightarrow \infty} e^{-t} u(t) \sim \phi(f). \tag{4.2}$$

5. **Further applications.** The problem of obtaining similar results for equations of the form

$$u'' - \left(1 + \frac{z}{f^2}\right)u = 0 \tag{5.1}$$

requires more detailed analysis. We shall discuss this in a subsequent paper, and also the equation

$$u'' + (1 + ze^{-\lambda t})u = 0, \tag{5.2}$$

and so on.

There is no difficulty in introducing the functions  $g_1(z)$  and  $g_2(z)$  defined by the relation

$$\lim_{t \rightarrow \infty} u(t)e^{-t} = c_1 g_1(z) + c_2 g_2(z), \tag{5.3}$$

where  $u$  is a solution of

$$u'' - (1 + f(t + z))u = 0, \quad u(0) = c_1, \quad u'(0) = c_2, \tag{5.4}$$

and obtaining similar linear differential equations, which are, of course, equivalent to the original equation (5.4). The real difficulty arises in obtaining initial values.

REFERENCES

1. R. Bellman, *Stability Theory of Differential Equations*, McGraw-Hill Book Company, Inc, New York, 1953
2. R. Bellman and J. M. Richardson, *On the asymptotic behavior of solutions of nonlinear differential equations*, to appear in this Quarterly