SINGULAR CASES IN THE OPTIMUM DESIGN OF FRAMES*

BY

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Abstract. Many previous studies in optimum design have determined the parameters so as to set equal to zero the first variation of the property to be optimized. The present paper shows that in some simple cases the minimum value may not be a point of zero variation and that points of zero variation may be relative maxima as well as minima. A general theoretical explanation of such behavior is given and applications are made to simple frames.

1. Introduction. The problem of optimum design has been the topic of numerous recent papers 1 to 18**. Several of these papers [1, 2, 5, 6], have been concerned with a structure with only a finite number of degrees of choice, such as a frame each of whose members must have a constant cross-section. Other papers investigate continuously varying beams and frames [15, 16], plates [3, 8, 11, 12, 13], and shells [4, 10, 14]. General theoretical results have been obtained in [7, 9, 17, 18].

Most of these studies have been confined to the problem of finding the minimum volume (or some quantity, such as weight, proportional to the volume). The present paper, however, is concerned with the general problem of minimization of a structural property without restriction to any particular property.

Specifically, we consider a beam or frame structure whose cross-section varies continuously. The safety of the structure is assumed to depend only on the magnitude of the bending moment and to be exhausted with the onset of plastic collapse.

Although a specific simple example is considered in Sec. 4, the aim of the present paper is to investigate general restrictions on the optimization of the structure, rather than to design any particular structure. It is shown that the critical feature in the mathematical problem which defines optimization is the order at which the optimizing property tends to zero (or to any other fixed value) as the bending moment tends to zero at discrete points or over finite portions of the structure. Specifically, it is shown that depending on the order of this zero the optimizing problem may exhibit a variety of methematical behavior varying from complete regularity with a unique solution corresponding to an analytical minimum to extreme irregularity with relative maxima and minima, non-analytic minima, and non-unique solutions. These types of behavior are illustrated in a simple example in Sec. 4. In conclusion, some inferences are drawn as regards the extension of these results to two and three dimensional structures.

2. Statement and formulation of the problem. We consider a beam or frame structure S whose shape, support conditions, and loading are given, but whose cross-section is given only to within a single parameter q. The problem of optimum design is to determine q at each point s of S so that a certain property of the structure is minimized, subject to the condition that the structure is safe.

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**For all numbered references, see bibliography.

Without loss of generality we shall refer to cost as the property to be minimized since a suitable definition of operating cost could require minimum weight or any other desirable property. Further, we shall assume that the cost is a fixed constant (which does not affect the optimization problem and so may be ignored) plus the integral over S of the unit cost, and that this latter is a function of the section parameter q. Thus we are ignoring any possible restriciton on the cost gradient, cost differences in welding different type sections, etc.

As safety criterion we require only that the structure be safe against plastic collapse, i.e. that the existing moment in each section be, in absolute value, equal to or less than the fully plastic moment. Since this moment is fully determined by the section, it is a function of q.

In order to make the optimization problem meaningful, we assume that the unit cost and the fully plastic moment are strictly monotonically increasing functions of q. If this is the case q can be eliminated from the problem and the unit cost F written as a strictly monotonically increasing function $F(M_p)$ of the fully plastic moment M_p. We further assume that $F(M_p)$ is positive, twice continuously differentiable (except, possibly, at the origin), strictly convex or concave ($F'' > 0$, $F'' < 0$ or $F'' = 0$ for all values of M_p), tends to infinity when M_p does, and is zero for $M_p = 0$. We note that this last restriction is unessential, for if $F(0) = F_0 \neq 0$, F_0 may be included in the already ignored constant and we may take the quantity $F(M_p) - F_0$ as unit cost.

A statically admissible state of stress is defined as any moment distribution which is in internal and external equilibrium with the loads P. If any statically admissible state of stress is taken to be fully plastic, q will be known at each point of the structure and a design thus defined. Its total cost, to within an additive constant, is

$$I\{M\} = \int_S F(|M(s)|)\, ds. \tag{2.1}$$

The optimization problem, then, is to determine a particular statically admissible state of stress $M^*(s)$ which minimizes I.

3. Analytical properties of the cost. It is well known (19) that all statically admissible states of stress arising in the structure S under the loads P can be expressed in the form

$$M(s) = M_0(s) + c_i M_i(s) \tag{3.1}$$

and that all states of stress of this form are statically admissible in the structure S under the loads P. In (3.1) the summation convention is understood to apply over the whole range $1, 2 \cdots, n$ of the repeated index i, n being the degree of redundancy of the structure; $M_0(s)$ is some arbitrary, statically admissible state of stress; $M_i(s)$ n linearly independent states of self stress in equilibrium with zero load; and c_i are n arbitrary constant coefficients.

With (3.1) introduced into (2.1) the cost becomes a function of the coefficients c_i and assumes the form

$$I(c_1, c_2, \cdots, c_n) = \int_S F(|M_0(s) + c_i M_i(s)|)\, ds. \tag{3.2}$$

The optimization problem is thus reduced to finding a set of values of the coefficients c_i which minimizes I. Before attempting its solution we shall study the analytical properties of the cost as a function of the variables c_i.

The expression (3.2) and the restrictions imposed on F lead to the immediate conclusion that I is a positive single valued function of the c_i.

It is not difficult to prove that I is also continuous and bounded for all bounded sets of values of the variables c_i and that it may grow beyond all bounds only if the absolute value of at least one of the variables does so.*

As regards differentiability, we may distinguish several cases depending on the order α at which the unit cost $F(M_p)$ tends to zero as the fully plastic moment M_p does so, and on whether the existing bending moment in the structure vanishes at discrete points or over finite portions of the structure.

(*i*) If $\alpha > 1$, the cost is continuously differentiable with respect to all the variables c_i, and for all sets of values of c_i. Its second differential (whether continuous or discontinuous, bounded or infinite) is always positive. The problem then admits of only one extremum which is an analytical minimum that can be determined by setting $dI = 0$.

(*ii*) If $0 < \alpha < 1$, the cost is continuously differentiable with respect to all the variables c_i for all sets of values of c_i for which the bending moment in the structure vanishes only at discrete points or not at all; however the cost may possess several relative maxima and minima. For the sets of values of the variables c_i for which the bending moment vanishes over finite portions of the structure, the cost possesses a non analytical relative minimum and its first derivatives exhibit an infinite discontinuity from $-\infty$ to $+\infty$.

(*iii*) If $\alpha = 1$, the cost is continuously differentiable with respect to all the variables c_i except for those sets of values of c_i for which the bending moment vanishes over finite portions of the structure. For these sets, the first derivatives with respect to the variables c_i exhibit a finite positive step discontinuity and the cost may possess a non-analytical minimum; but for no sets of values of the variables does it possess a relative maximum.

(*iv*) The possibility $\alpha \leq 0$ has been excluded by the assumption that $F(M_p)$ is zero for $M_p = 0$ and strictly monotonically increasing.

It is evident from the above statements that methods based upon finding locations where $dI = 0$ may not be effective in finding the minimum value of I in some cases where $0 < \alpha \leq 1$. Indeed, such a procedure would conceivably lead to no solution at all, to a relative maximum or to a relative minimum which is not the absolute minimum. That these types of undesirable behavior may actually occur in simple cases, is shown in the following section.

4. Example. As an example of the types of behavior discussed in the preceding section we consider the frame in Fig. 1a. It is only once statically indeterminate and as redundant we choose the horizontal reaction at the hinged supports; it will be indicated by c_1. There exists only one state of self stress as shown in Fig. 1b; the cost is a function of only one variable and its graph can be drawn easily to exhibit the features described in the previous section. M_0 is depicted in Fig. 1c, and Figs. 1d, e, f are representative of the statically admissible states of stress that obtain in the structure when the redundant lies, respectively, in the regions

$$c_1 \leq -\lambda\mu P, \qquad (d)$$

$$-\lambda\mu P \leq c_1 \leq 0, \qquad (e)$$

*Precise statements and complete proofs of the analytical properties of the cost are to be found in the Appendix A.

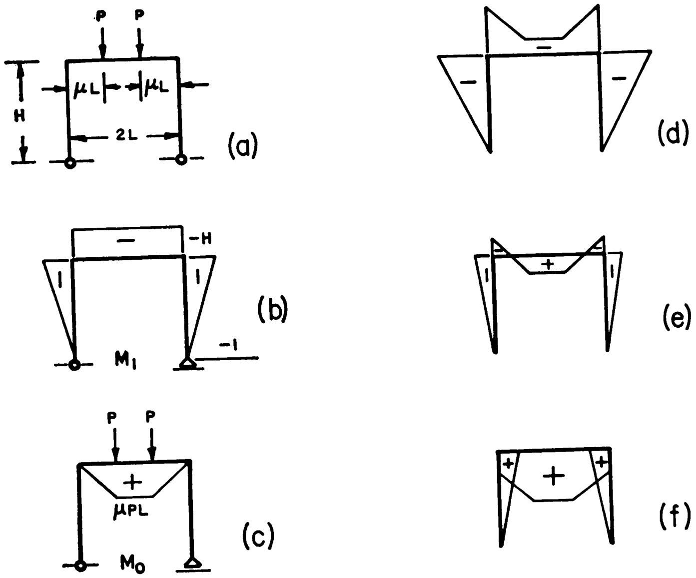


FIG. 1. Portal Frame under two Loads. (a) Loaded Frame; (b) State of Self Stress; (c) Basic Statically Admissible State of Stress; (d), (e), (f) Typical Moment Distributions.

$$0 \leq c_1, \qquad (f)$$

λ being the ratio L/H.

As a simple class of functions which exhibit all indicated types of behavior, we choose

$$F(M_p) = (M_p)^\alpha, \qquad 0 < \alpha \leq 1. \tag{4.1}$$

We find it convenient to introduce dimensionless quantities

$$\eta = \tfrac{1}{2}(\alpha + 1)I/(P^\alpha L^{\alpha+1}), \qquad \xi = c_1/(\lambda P). \tag{4.2}$$

It then follows easily from Eqs. (3.2), (4.1) and Fig. 1 that the dimensionless cost and its first derivative are

$$\begin{aligned}
\eta &= (-\xi)^{\alpha+1} - (-\mu - \xi)^{\alpha+1} + (\alpha + 1)(1 - \mu)(-\mu - \xi)^\alpha + (-\xi)^\alpha/\lambda, \\
d\eta/d\xi &= -(\alpha + 1)[(-\xi)^\alpha - (-\mu - \xi)^\alpha + \alpha(1 - \mu)(-\mu - \xi)^{\alpha-1}] \\
&\qquad - \alpha(-\xi)^{\alpha-1}/\lambda, \quad \text{for} \quad \xi \leq -\mu; \\
\eta &= (-\xi)^{\alpha+1} + (\mu + \xi)^{\alpha+1} + (\alpha + 1)(1 - \mu)(\mu + \xi)^\alpha + (-\xi)^\alpha/\lambda, \\
d\eta/d\xi &= -(\alpha + 1)[(-\xi)^\alpha - (\mu + \xi)^\alpha - \alpha(1 - \mu)(\mu + \xi)^{\alpha-1}] \\
&\qquad - \alpha(-\xi)^{\alpha-1}/\lambda, \quad \text{for} \quad -\mu \leq \xi \leq 0; \\
\eta &= -(\xi)^{\alpha+1} + (\mu + \xi)^{\alpha+1} + (\alpha + 1)(1 - \mu)(\mu + \xi)^\alpha + (\xi)^\alpha/\lambda, \\
d\eta/d\xi &= -(\alpha + 1)[(\xi)^\alpha - (\mu + \xi)^\alpha - \alpha(1 - \mu)(\mu + \xi)^{\alpha-1}] \\
&\qquad + \alpha(\xi)^{\alpha-1}/\lambda, \quad \text{for} \quad 0 \leq \xi.
\end{aligned}$$

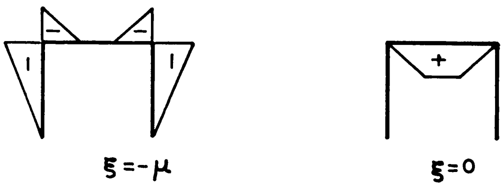


FIG. 2. Singular States of Stress in Portal Frame.

From the expressions for the derivative it is immediately obvious that at the points $\xi = -\mu$ and $\xi = 0$ a step discontinuity will appear in the slope of η. If $\alpha = 1$, this step discontinuity will be $2\alpha(\alpha + 1)(1 - \mu)$ at the first point and $2\alpha/\lambda = 2/\lambda$ at the second; if $\alpha < 1$ it will be from $-\infty$ to $+\infty$ at both points. A kink or cusp will therefore develop at each of these points; they are due to the vanishing of the bending moment between the loads or along the columns, respectively. The states of stress which produce these singularities are shown in Fig. 2. The kinks and cusps can be seen in Figs. 3 and 4.

In Fig. 3 $\mu = 0.8$, $\alpha = 1$; the graphs of η have been drawn for the values of $\lambda = 0.3$, 0.5, 0.7. For $\lambda \leq 0.5$ the kink at $\xi = 0$ represents the optimum solution; it leads to the unrealistic design of vanishing column section and a beam acting as simply supported. As λ increases beyond the value of 0.5 the minimum occurs at a regular point gradually moving away from $\xi = 0$. The value of this minimum becomes gradually smaller than $\eta(0) = 0.96$ and the corresponding designs gradually transfer material from the beam to the columns.

In Fig. 4 $\mu = 0.8$ and $\alpha = \frac{1}{2}$; the graphs of η have been drawn for the values of $\lambda = 1$, 1.585, 2.151, 4.167. In all cases there exist more than one relative extrema. For $\lambda < 2.151$ the governing minimum is the cusp at $\xi = 0$; it leads to the same unrealistic design as before: vanishing columns and the beam acting as simply supported. For $\lambda > 2.151$ the governing minimum is a regular one occurring at a point ξ between zero and

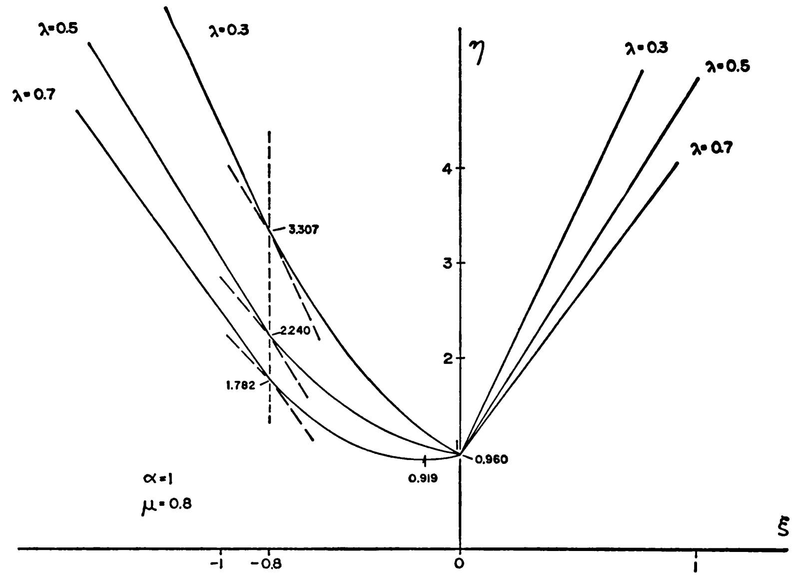


FIG. 3. Non Dimensional Cost Curves of Portal Frame for $\alpha = 1$.

$-\mu = -0.8$, where the slope of η is zero. For $\lambda = 2.151$ there exist two distinct solutions with the same minimum cost $\eta(0) = \eta(-0.3175) = 0.984$. All values of ξ between these two yield a bigger η and can never lead to an optimum design, whatever the value of λ may be. The transition here from one mode of design (vanishing columns, simply supported beam) to the other is abrupt. It is also important to note that for $\lambda < 1.585$ the cost exhibits two relative minima and one maximum; for $\lambda > 1.585$ there exist three relative minima and two maxima. When seeking the optimum solution, therefore, it is not sufficient to find a relative minimum; one must also prove that this is the governing one. Furthermore in both cases $\alpha = 1$ and $\alpha < 1$, minima do not occur at regular stationary points, where the derivatives are zero; they may be found at singular points and such minima may be governing for a wide range of variation of the parameters of the problem.

5. Conclusions. It has been shown generally, for a wide class of cases, and illustrated by a simple example, that the problem of optimum design of a beam or frame structure with continuously varying cross-section is a highly complicated one. Only when $\alpha > 1$ will the solution of the problem always be represented by the unique analytical minimum of a regular convex function. When $0 < \alpha < 1$, the cost may exhibit relative maxima and minima, as well as non analytical minima at which the first differential is discontinuous or even unbounded. The non analytical minima represent critical modes of design which may persist for a range of variation of the shape or the loading of the structure and then give suddenly way to fundamentally different designs. At the transitory stage, more than one distinct design may exist for the minimum cost; in no sense, therefore, can uniqueness of solution be claimed. In the intermediate case $\alpha = 1$ the behavior is less irregular, and procedures are available [16] for finding the absolute minimum. However, since the optimum design may be neither analytic nor unique, the powerful methods of variational calculus are not generally applicable.

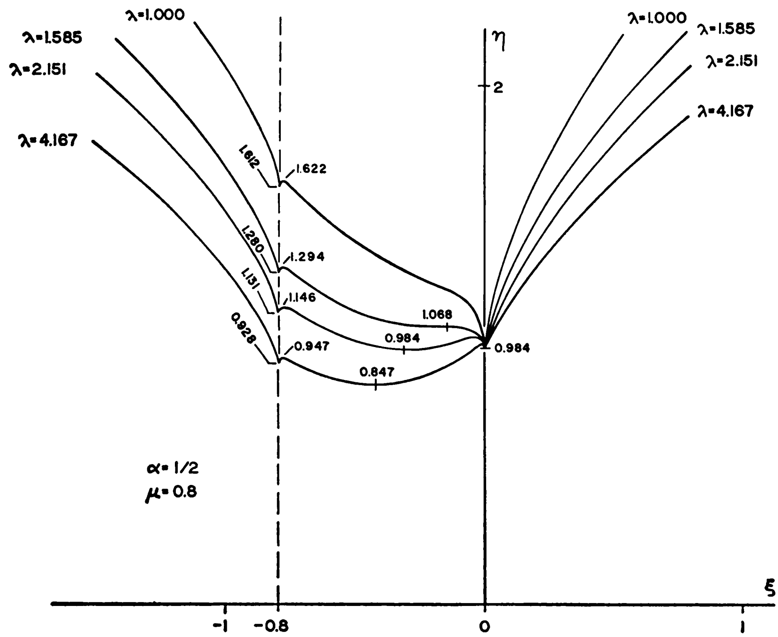


Fig. 4. Non Dimensional Cost Curves of Portal Frame for $\alpha = \frac{1}{2}$.

That the range $0 < \alpha \leq 1$ is not unimportant is indicated by the fact that most minimum volume designs lead to values in this range. Thus if the shape is given, $F = B_1(M_p)^{2/3}$; for a rectangle of given width $F = B_2(M_p)^{1/2}$; for a rectangle of given height or an ideal sandwich or I section, $F = B_3 M_p$ where B_1, B_2, B_3 are constants depending on the yield stress and the constant parameters of the section.

It is interesting to note that two recent papers [17, 18] show that some of the above features are also exhibited by two dimensional structures, plates and shells. Although their approach is different it seems that minimum volume design of a sandwich section according to the Tresca yield criterion corresponds to the case $\alpha = 1$ of the present paper and that minimum volume design of a solid plate according to the same conditions corresponds to $\alpha = \frac{1}{2}$.

Appendix A

Proof of the Analytical Properties of the Cost

We recall that we have taken the unit cost $F(M_p)$ to be a positive, strictly monotonically increasing function of the absolute value of the bending moment, twice continously differentiable (except possibly at the origin where $F(0) = 0$) and either strictly convex ($F'' > 0$) or strictly concave ($F'' < 0$) or linear ($F'' = 0$).

Under these conditions it is easy to show that the cost is a continuous function in all the variables c_i, by simply extending the theory of integrals depending on one parameter as given in standard textbooks, for instance [20].

Considering next those sets of values of c_i for which the bending moment does not become zero anywhere in S (although it may change sign at points of step discontinuity) it can be shown in the same way that, for these sets of values of c_i, the cost is a differentiable function and that

$$\partial I/\partial c_i = \sum_{r+} \int_{Sr+} F'(|M(s)|)M_i\, ds - \sum_{r-} \int_{Sr-} F'(|M(s)|)M_i\, ds, \tag{A.1}$$

$$d^2 I = (\partial^2 I/\partial c_i \partial c_k)\, dc_i\, dc_k = \int_S F''(|M(s)|)M_i M_k\, dc_i\, dc_k\, ds. \tag{A.2}$$

where S_{r+} denote the portions of the structure over which $M(s) > 0$ and S_{r-} those over which $M(s) < 0$. As $M_i M_k dc_i dc_k$ is a non-negative quadratic form, the cost is also convex or concave in the same sense as the unit cost.

Consider next those sets of values for which the bending moment is zero but only at a finite number of discrete points s_p of S. If the structure is composed of straight elements and the load of concentrated forces, the order of zero at these points will be one.*

If each s_p is enclosed in a small interval L_p, equations (A.1) and (A.2) will still be applicable to the part of the structure lying outside these intervals. Inside each of them-provided its length is chosen adequately small-the integrand of I can be approximated to any desired degree of accuracy by an expression of the form

$$F(|M(s)|) = |a + c_i a_i + (b + c_i b_i)s|^\alpha$$

where a, a_i, b, b_i are constants. Taking, without loss of generality, the left end of the interval as the origin of s and $a + c_i a_i > 0$ (hence $b + c_i b_i < 0$, Fig. 5a) we obtain for

*The effect of a zero of a higher order is discussed in Appendix B the contribution of L_p to the index and its derivatives

$$I_p = -\{|a + c_i a_i + (b + c_i b_i)L_p|^{\alpha+1} + |a + c_i a_i|^{\alpha+1}\}/(\alpha + 1)(b + c_i b_i), \tag{A.3}$$

$$\begin{aligned} \partial I_p/\partial c_j &= b_j\{a + c_i a_i + (b + c_i b_i)L_p|^{\alpha+1} + |a + c_i a_i|^{\alpha+1}\}/(\alpha + 1)(b + c_i b_i)^2 \\ &\quad + \{(a_j + b_j L_p)\cdot |a + c_i a_i + (b + c_i b_i)L_p|^{\alpha} - a_j\, |a + c_i a_i|^{\alpha}\}/(b + c_i b_i), \end{aligned} \tag{A.4}$$

$$\begin{aligned} \partial^2 I_p/\partial c_j \partial c_k &= -2b_j b_k\{|a + c_i a_i + (b + c_i b_i)L_p|^{\alpha+1} + |a + c_i a_i|^{\alpha+1}\}/(\alpha + 1)(b + c_i b_i)^3 \\ &\quad - 2b_j\{(a_j + b_j L_p)\, |a + c_i a_i + (b + c_i b_i)L_p|^{\alpha} - a_j\, |a + c_i a_i|^{\alpha}\}/(b + c_i b_i)^2 \\ &\quad - \alpha\{(a_j + b_j L_p)(a_k + b_k L_p)\, |a + c_i a_i + (b + c_i b_i)L_p|^{\alpha-1} \\ &\quad + a_j a_k\, |a + c_i a_i|^{\alpha-1}\}/(b + c_i b_i). \end{aligned} \tag{A.5}$$

Let now $L_p \to 0$ as $b + c_i b_i$ remains constant; then also $a + a_i c_i \to 0$ and $a + c_i a_i + (b + c_i b_i)L_p \to 0$.

(*i*) If $\alpha \geq 1$, $\alpha + 1$ and α are positive and $\alpha - 1$ non-negative; all terms in the expressions (A.3), (A.4) and (A.5) tend to zero and the contribution of each L_p can be made arbitrarily small while the expressions (A.1) and (A.2) are applicable over as large a portion of the structure as required. At the limit they are valid over the whole structure yielding finite and continuous values for the cost and its first and second derivatives.

(*ii*) If $0 < \alpha < 1$, $\alpha + 1$ and α are still positive but $\alpha - 1$ is negative; the previous argument is applicable only to the first derivatives which, thus, are still finite and continuous and can be obtained from (A.1). The second derivatives however are neither continuous nor can they be obtained from (A.2); we may gain some information about their behavior by taking the interval L_p fixed and varying c_i so that the zero point of the bending moment disappears at the right hand end of the interval (Fig. 5b). The significant terms for this case are

$$\begin{aligned} \partial^2 I_p/\partial c_j \partial c_k \sim \alpha\{(a_j + b_j L_p)(a_k + b_k L_p)\, |a + c_i a_i + (b + c_i b_i)L_p|^{\alpha-1} \\ + a_j a_k\, |a + c_i a_i|^{\alpha-1}\}/(b + c_i b_i). \end{aligned} \tag{A.6}$$

Comparing (A.5) and (A.6) we see that, as the zero of the bending moment appears at the right end of the interval L_p, the second derivatives jump from $-\infty$ to $+\infty$ (Fig. 5c), or vice versa if $b + c_i b_i > 0$. From then onwards they remain finite, but may change sign or develop again infinite discontinuities as the bending moment tends to zero at other points of the structure. The cost therefore may exhibit oscillations and hence have more than one relative extrema; each extremum is analytical and can be obtained by setting the first derivatives equal to zero.

Finally consider the sets of values c_i for which the bending moment is zero over finite portions of the structure. An adequately small variation Δc_j will then produce such a small bending moment over parts or the whole of the above portions, say S_{r_0}, that the cost variations can be expressed, with adequate accuracy as*

$$\Delta I = \sum_{r_0} \int_{S_{r_0}} |\Delta c_j M_j|^{\alpha}\, ds = |\Delta c_j|^{\alpha} \sum_{r_0} \int_{S_{r_0}} |M_j|^{\alpha}\, ds.$$

*The summation convention does not apply to the rest of this section.

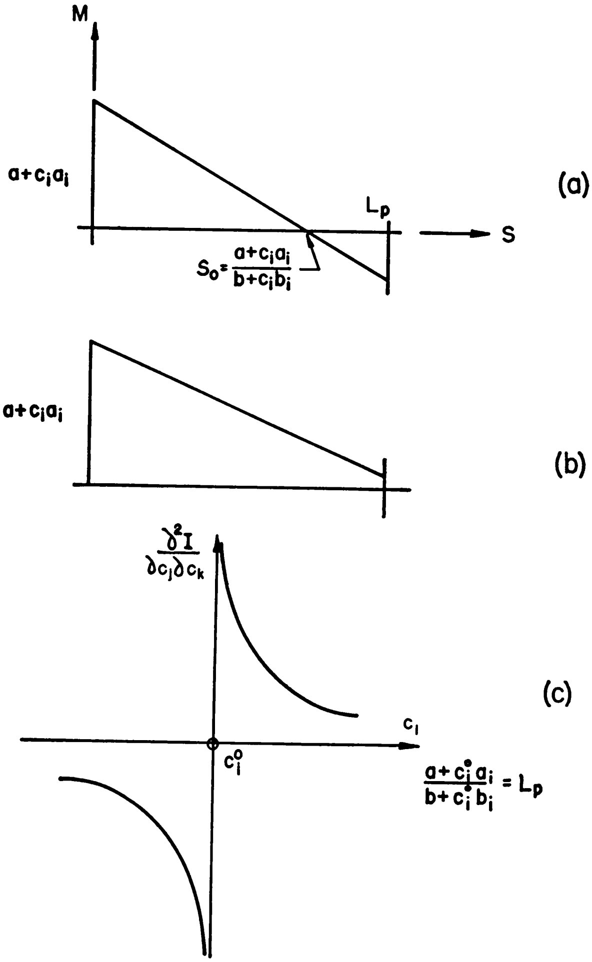


FIG. 5. Second Variation of the Cost. (a) State of Stress after Occurrence of Zero; (b) State of Stress before Occurrence of Zero; (c) Second Derivative of the Cost as Function of a Redundant.

The first derivative then is

$$\frac{dI}{dc_i} = \lim \frac{|\Delta c_i|^\alpha}{\Delta c_i} \sum_{r_0} \int_{S_{r_0}} |M_i|^\alpha \, ds \quad \text{as} \quad \Delta c_i \to 0.$$

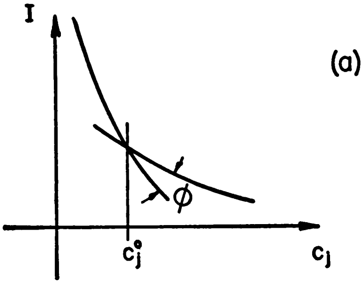
If $\alpha > 1$, then $dI/dc_i = 0$; if $\alpha = 1$, then dI/dc_i shows a positive finite jump,

$$\phi = 2 \sum_{r_0} \int_{S_{r_0}} |M_i|^\alpha \, ds \qquad \text{(Fig. 6a)};$$

if $\alpha < 1$, then dI/dc_i jumps from $-\infty$ to $+\infty$ (Fig. 6b).

It is evident, in the two last cases the cost may exhibit a nonanalytical extremum which has to be a minimum.

Classifying, finally the above results with respect to the values of α we obtain the statements of Section 3.



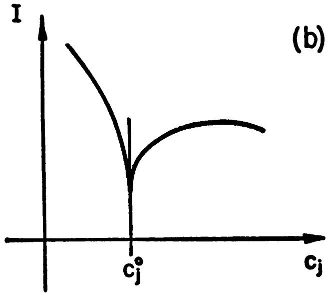


FIG. 6. Discontinuities in the First Variation of the Cost. (a) For $\alpha = 1$; (b) For $0 < \alpha < 1$.

APPENDIX B

EFFECT OF THE ORDER OF ZERO ON THE DIFFERENTIABILITY OF THE COST

If the order at which the bending moment becomes zero at discrete points of the structure is $\beta > 1$, then the expression for the unit cost in a small interval around this point will be of the form

$$F(|M(s)|) = |A + Bs^\beta|^\alpha.$$

The integrals in (A.1) may then be divergent when

$$(1 - \alpha)\beta \geq 1. \tag{B.1}$$

The approach used in Appendix A, however, is no longer practicable as the integrations which led to the expressions (A.3) (A.4) and (A.5) do yield transcendental rather than algebraic functions. Although it can be shown that even in the case when the first derivatives become unbounded they are continuous, in the sense that they tend to the same infinite limit on both sides of the point of zero bending moment, we will restrict ourselves to illustrating this case by a simple example as it does not introduce a possibility of an extremum.

Consider a fixed end beam under uniformly distributed load (Fig. 7a). Figs. 7b, c show a state of self stress M_1 and a statically admissible state of stress M_0. Figs. 7d, e, f are representative of the possible states of stress. Taking $\alpha = \frac{1}{2}$ and using dimensionless quantities

$$\xi = 2c/(wL^2), \qquad \eta = I\sqrt{2}/L^2\sqrt{w},$$

we obtain for these three types of stress, respectively

$$\eta = (-\xi)^{1/2} + (\xi + 1) \quad \log \quad [(-\xi - 1)^{1/2}/(1 + (-\xi)^{1/2})], \qquad \xi < -1;$$

$$\eta = (-\xi)^{1/2} + (\xi + 1) \quad \log \quad [(\xi + 1)^{1/2}/(1 + (-\xi)^{1/2})] + \tfrac{1}{2}\pi(\xi + 1), \qquad -1 < \xi < 0;$$

$$\eta = (\xi)^{1/2} + (\xi + 1) \quad \arcsin \quad (1/(\xi + 1)^{1/2}), \qquad 0 < \xi.$$

These results are plotted in (Fig. 8). For the value $\xi = -1 (c = -wL^2/2)$ the state of

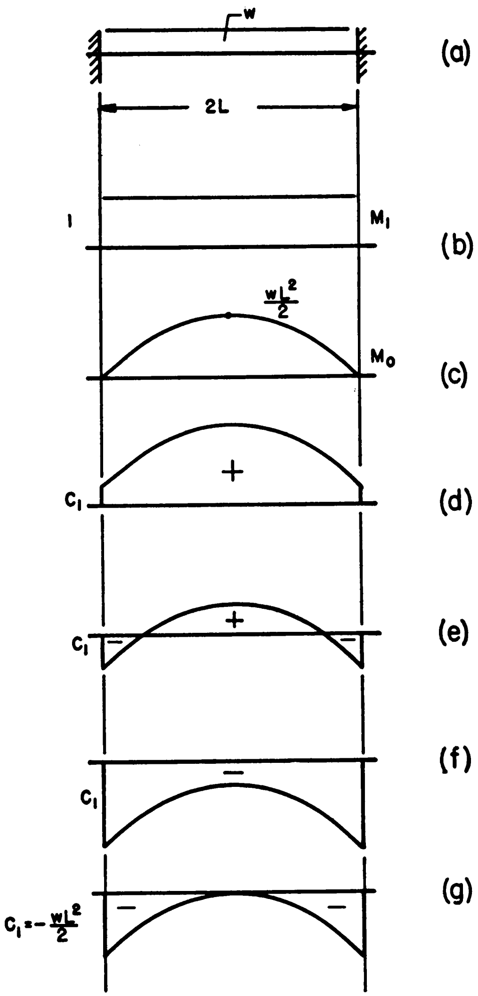


FIG. 7. Fixed End Beam under Uniformly Distributed Load. (a) Loaded Beam; (b) State of Self Stress; (c) Basic Statically Admissible State of Stress; (d), (e), (f) Typical Moment Distributions; (g) Singular Moment Distribution.

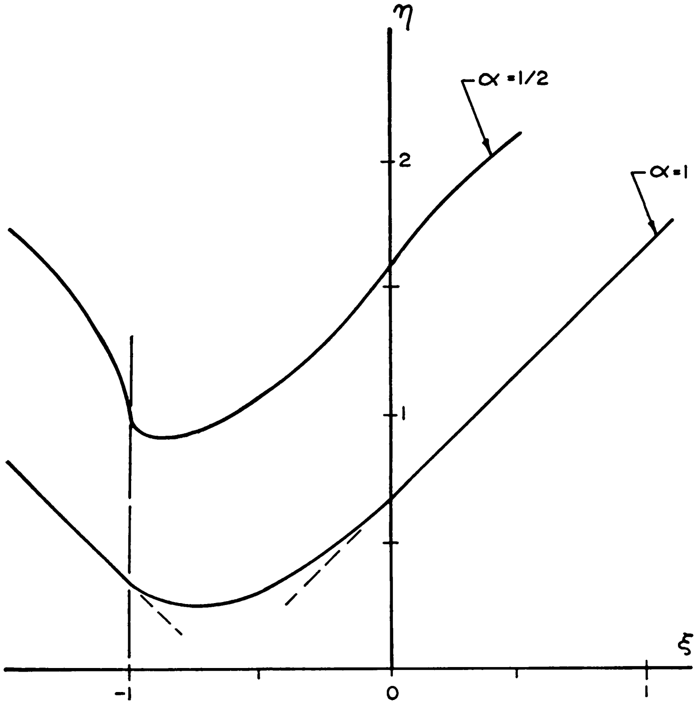


FIG. 8. Non Dimensional Cost Curves of Fixed End Beam.

stress has a zero of order 2 (Fig. 7g) and the graph of η shows an infinite slope as anticipated by the criterion (B.1)

$$(1 - \alpha)\beta = (1 - \tfrac{1}{2})\cdot 2 = 1.$$

There is, however, no extremum at this point.

For the sake of comparison the curve of η for $\alpha = 1$ is also shown in Fig. 7; as anticipated it shows no singularities of slope.

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