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A VARIATIONAL PRINCIPLE FOR NONLINEAR NETWORKS*

By GARRETT BIRKHOFF (*Harvard University*)

It is the main purpose of this note to describe a variational principle for nonlinear networks of the kind studied by J. B. Diaz and the author in [1]. It will then be shown that this principle is closely related to an electromechanical analogy discovered by Duffin [2], in connection with n-dimensional mass-spring networks.

The notation and terminology of [1] will be used below. That is, a connected network (oriented graph) N of n nodes A_k and r *links* a_i will be assumed defined by an *incidence matrix* $\| \epsilon_{ki} \|$, where ϵ_{ki} is $+1$, -1, or 0 when A_k is the initial node, the final node, or not incident on a_{ii}, respectively. In N, a subset ∂N of *boundary nodes* is distinguished, and the set of all links is denoted by L.

It is assumed that, in *equilibrium*, the current $i_i = c_i(\Delta u_i)$ is a specified continuous *increasing* function $c_i(\Delta u_i)$ of the potential drop $\Delta u_i = \sum_k \epsilon_{ki} u_i$ across the j-th link. At each interior node A_k (not on ∂N), the usual node law $\sum_{j=1}^r \epsilon_{ki} i_i = 0$ is assumed. At each boundary node A_h, it is assumed that either u_h is known or that (on the residual set $\partial^* N$)

$$\sum_i \epsilon_{hi} i_i = \sum_i \epsilon_{hi} c_i \left(\sum_k \epsilon_{ki} u_k \right) = F_h(u_h) \quad \text{on} \quad \partial^* N, \tag{1}$$

where $F_h(u)$ is a continuous *non*-increasing function of u. (The case $F_h(u) = \nu_h$ constant leads to the Neumann problem.)

With the preceding problem, we now associate the function

$$V = \sum_L V_i(\Delta u_i) + \sum_{\partial^* N} W_h(u_h), \tag{2}$$

where the functions V_i and W_h are defined as the indefinite integrals

$$V_i(\Delta u_i) = \int_0^{\Delta u_i} c_i(s)\, ds, \qquad W_h(u_h) = -\int_0^{u_h} F_h(s)\, ds. \tag{2'}$$

We take as unknowns the values $u_k = u(A_k)$ at those interior and boundary nodes where the potential u_k is *not* given.

An easy computation shows that the first variation of the expression (2) defined by (2') is

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$$\delta V = \sum_L c_i(\Delta u_i) \sum \epsilon_{ki} \, \delta u_k - \sum_{\partial \cdot N} F_h(u_h) \, \delta u_h \,, \tag{3}$$

where the variations δu_h are independent over the nodes where u_h is not given.

THEOREM 1. The conditions for equilibrium are equivalent to the variational equation $\delta V = 0$.

Proof. At a boundary node where u_h is not given, the coefficient of δu_h in (3) vanishes if and only if

$$\sum_{L_h} \epsilon_{hi} i_i = F_h(u_h), \tag{4}$$

summed over the links incident with A_h. Clearly, equations (4) and (1) are identical. The proof for internal nodes is identical.

The preceding proof makes use only of the continuity of the functions c_i and F_h, which makes valid the differentiation of (2) to get (3). Therefore, the statement of Theorem 1 is true whether or not the c_i are increasing and the F_h non-increasing functions.

THEOREM 2. If the functions c_i are non-decreasing and the F_h non-increasing, then the function $V(\mathbf{u})$ defined by (2)–(2') is *convex*.

Proof. It suffices to prove that

$$V(\mathbf{u} + \mathbf{v}) + V(\mathbf{u} - \mathbf{v}) - 2V(\mathbf{u}) \geqq 0. \tag{5}$$

But, referring to the definition (2)–(2'), we have

$$\Delta^2 V_i = V_i(\Delta u_i + \Delta v_i) + V_i(\Delta u_i - \Delta v_i) - 2V_i(\Delta u_i)$$

$$= \int_0^{\Delta v_i} [c_i(\Delta u_i + s) - c_i(\Delta u_i - s)] \, ds \geqq 0$$

for all j, provided the c_i are non-decreasing. Likewise,

$$\Delta^2 W_h = W_h(u_h + v_h) + W_h(u_h - v_h) - 2W_h(u_h)$$

$$= -\int_0^{\Delta u_h} [F_h(u_h + s) - F_h(u_h - s)] \, ds \geqq 0,$$

provided every F_h is non-increasing.

It is interesting that the statement of Theorem 2 is true whether or not the c_i and the F_h are continuous. It is also interesting to inquire into the conditions for strict convexity: that strict inequality holds in (5). Inspecting the inequalities used in proving (5), we see

THEOREM 3. The function $V(\mathbf{u})$ defined by (2)–(2') is *strictly* convex if and only if the c_i are increasing functions and the F_h decreasing functions.

COROLLARY. Under the preceding hypotheses, there is at most one equilibrium solution of the nonlinear network problem, and this occurs when V has a strict minimum.

For the existence of an absolute minimum to a (continuous) convex function $V(\mathbf{u})$, it is sufficient that $\lim_{|\mathbf{u}| \to \infty} V(\mathbf{u}) = +\infty$. Therefore, it is sufficient that all functions $V_i(\Delta u_i)$ and $W_h(u_h)$ should tend to $+\infty$ with $|\Delta u_i|$, as this will make some terms in (2) tend to $+\infty$ with $|\mathbf{u}|$, while the others remain bounded below. We conclude

THEOREM 4. If the functions c_i are non-decreasing and the F_h non-increasing, *and* if these functions assume both signs, then V has a minimum.

Note that, since monotone functions are always integrable, continuity is not assumed above.

COROLLARY. If the c_i and F_h are continuous in Theorem 4, then the network problem has a solution.

In conclusion, it may be of interest to describe the mechanical analog of the network problem, whose potential (strain) energy function is $V(\mathbf{u})$. In this analog, the nodes are represented by smooth *rods*, whose displacements from given positions are the u_i . These rods are constrained to slide along the u-axis, a constraint which can be imposed by inserting the rods in a long smooth tube. These rods can then be imagined as joined by one or more *springs*, for each link a_i whose total *stress* $c_i (\Delta u_i)$ depends on the relative displacement of the rods joined by the link a_i . This acts equally and in opposite directions as a force on these rods, as prescribed by the ϵ_{ki} (thus *stress* or *force* is the mechanical analog of *current*). At internal nodes, the condition for equilibrium is that the sum (resultant) of the forces be zero. At boundary nodes, one can suppose externally attached springs exert a force F_h which depends on the displacement of the rod A_h .

This model is very similar to that suggested by Duffin [2], provided the junction points are constrained to move in one dimension. However, Duffin's "force functions" $f_{ij}(| \mathbf{r}_i - \mathbf{r}_j |)$ were assumed to be defined only for non-negative arguments, to satisfy $f(0) = 0$, and to be unbounded. This would correspond roughly to assuming the $c_i(\Delta u_i)$ to be *odd* functions tending to infinity with $| \Delta u_i |$, and these assumptions seem to be unnecessary for the existence and uniqueness theorems stated above. Duffin also considers only the Dirichlet problem (all F_h vanish).

Duffin's hypotheses are, of course, appropriate for networks of springs under tension in more than one dimension.

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AN INTEGRAL EQUATION OCCURRING IN PLASMA OSCILLATIONS*

By R. S. B. ONG (*The University of Michigan*)

The stability of single hump velocity distributions in a collisionless hot plasma without magnetic field is usually shown by considering the linearized Landau-Vlasov self-consistent set of equations as an initial value problem. The Laplace—or the one-sided Fourier transform with respect to the time variable is then commonly used in the analysis. In this method, one needs the analytic continuation of the function considered beyond its original domain of definition in order to evaluate its inverse transform by complex contours. This introduces a certain degree of artificiality in the analysis. The purpose of this article is to show that one can deduce the stability of single hump velocity distributions by means of an argument which avoids this difficulty.

The one-dimensional linearized problem of longitudinal oscillations of a collisionless hot plasma without an external magnetic field is described by the following well-known self-consistent set of equations: