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ON VARIATIONAL PRINCIPLES IN THERMOELASTICITY AND HEAT CONDUCTION*

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Abstract. The variational principles for displacements, for stresses and for both displacements and stresses in isothermal elasticity are extended to the coupled processes of thermoelasticity and heat conduction in a three-dimensional, anisotropic body. The character of these principles is examined and it is established that in a stable system one is concerned with a minimum, a maximum and a stationary value problem, respectively.

1. Introduction. In the theory of elasticity variational principles have proved to be of great value both in establishing the foundations of the theory as well as in applying numerical procedures for the solution of specific problems. As examples, it might suffice to mention here the use of variational principles for the proof of uniqueness, and Ritz's method for the numerical solution of boundary value problems. Furthermore, and particularly in recent years, variational principles have been used extensively in deriving approximate two- and one-dimensional theories for three dimensional problems.

A concise description of various aspects of variational principles was presented in a recent study by Reissner [1]. In this same paper, Reissner has shown that the well known minimum principle for displacements (due to Green) and the maximum principle for stresses (due to Castigliano) are direct consequences of his own more general stationary value principle for both stresses and displacements.

In thermoelasticity and heat conduction, Biot [3] formulated recently a variational principle, which yields the thermoelastic equilibrium equations and the heat conduction equation as the Euler differential equations. A variational principle complementary to Biot's procedure was established by the present writer [4]. It was shown in a later investigation [5], for the special case of a one-dimensional body and particular boundary conditions, that it is possible to extend Green's Castigliano's and Reissner's variational theorems to thermoelasticity and heat conduction. The nature of these principles, however, was not investigated in [5].

The purpose of the present paper is to formulate these principles for an anisotropic three-dimensional medium and for a more general class of boundary conditions. Furthermore, the character of these variational principles is here examined, and it is found that it is completely preserved in the passage from isothermal elasticity to thermoelasticity and heat conduction, provided Biot's thermoelastic potential and dissipation function are positive definite. As a consequence of the second law of thermodynamics, the dissipation function has to satisfy this condition. The positive-definiteness of the thermoelastic potential, however, is not required by any law but may be simply taken as definition of a stable system.

In establishing these principles, following Biot, the kinematic variables are taken to be the solid displacement and the entropy displacement. The dynamic variables are the stress, which has to be resolved into the isothermal part and a thermal part, the tempera-

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ture increment above a reference temperature, and a force, which is conjugate to the entropy displacement. Biot had not introduced this force in thermoelasticity, but he did so in formulating a series of variational principles in irreversible thermodynamics [6], [7]. Following his terminology, this force might be called a thermal dis-equilibrium force. The details of proofs, which are analogous to Reissner's [1], are omitted here for the sake of brevity.

It should be noted that all of the following is equally applicable to separate (uncoupled) problems of thermoelasticity and heat conduction, and also to elasticity of an elastic porous solid, whose pores are filled with a compressible viscous fluid, using the analogy established by Biot [4].

2. The boundary value problem. The equations of linear thermoelasticity and heat conduction for a body occupying a volume V bounded by a surface $A = A_u + A_t = A_s + A_0$, using in part the formulations given in references [3], [4], consist of the following set:

in V:
$$\tau_{ij,j} - \beta_{ij}\theta_{,j} + \psi_{,u_i} = 0 \tag{2.1a}$$
$$\theta_{,i} - g_i = 0 \tag{2.1b}$$
$$\gamma_{ij} - W_{,\tau_{ij}} = 0 \tag{2.1c}$$
$$\gamma + W_{,\theta} = 0 \tag{2.1d}$$
$$s_i + D_{,g_i} = 0 \tag{2.1e}$$

on A_u:
$$u_i = \Phi_{,t_i} \tag{2.2a}$$

on A_s:
$$s_i = \Sigma_{,\theta_i} \tag{2.2b}$$

on A_t:
$$t_i = X_{,u_i} \tag{2.2c}$$

on A_θ:
$$\theta_i = \Theta_{,s_i} \tag{2.2d}$$

Use is made of the summation convention. A comma preceding a subscript indicates differentiation with respect to that variable, except that i (or j) after a comma stands for differentiation with respect to the Cartesian coordinate x_i (or x_j).

In the above equations $\tau_{ij} = \tau_{ji}$ denotes the components of stress if the temperature increment θ above a reference absolute temperature T_r vanishes. The total stress, in general, is thus $\tau_{ij} - \beta_{ij}\theta$, where β_{ij} is related to the thermal dilatation properties of the material. Further, u_i and s_i indicate the components of solid and entropy displacement, respectively. It is recalled that the entropy displacement s_i is related to the entropy s by the equation

$$s = -s_{i,i}$$

The elastic strain is defined as

$$\gamma_{ij} = (u_{i,j} + u_{j,i})/2 \tag{2.3}$$

while γ denotes the "thermoelastic dilatation" defined as

$$\gamma = s_{i,i} + \beta_{ij}u_{i,j} \tag{2.4}$$

The body force function ψ is taken in the form

$$\psi = X_i u_i \tag{2.5}$$

where the X_i are given functions of the coordinates.

W is Biot's thermoelastic potential, expressed here as a quadratic form of isothermal components of stress and the temperature increment

$$W = \tfrac{1}{2}B_{ijkl}\tau_{ij}\tau_{kl} + \tfrac{1}{2}c\theta^2/T_r \tag{2.6}$$

c is the specific heat per unit of volume for zero strain. D is Biot's dissipation function, expressed here as a quadratic form of the "thermal dis-equilibrium force" g_i, the quantity conjugate to the entropy displacement s_i

$$D = \frac{k_{ij}}{pT_r} g_i g_j \tag{2.7}$$

k_{ij} is the thermal conductivity tensor and p is the time operator $\partial/\partial t$.

The material constants must obey the following symmetry properties

$$\beta_{ij} = \beta_{ji} \tag{2.8a}$$

$$B_{ijkl} = B_{jikl} = B_{ijkl} = B_{klij} \tag{2.8b}$$

$$k_{ij} = k_{ji} \tag{2.8c}$$

t_i is the x_i-component of the total surface traction given by

$$t_i = p_i - \beta_{ij}\theta_j = \cos(n, x_i)(\tau_{ij} - \beta_{ij}\theta) \tag{2.9}$$

where n is the outward drawn normal to the surface A. The notation is used here

$$\theta_i = \theta \cos(n, x_i) \tag{2.10}$$

The total boundary surface A is divided into two parts in two different ways. One decomposition is such that $A = A_u + A_t$ and u_i is prescribed on A_u and t_i is prescribed on A_t. The other decomposition is such that $A = A_s + A_\theta$ and s_i is prescribed on A_s and θ is prescribed on A_θ.

The formulation of the boundary conditions (2.2) does not include the more general case in which any factor in each of the six products $u_i t_i$, $s_i\theta_i$ may be specified at each point of the total boundary A. Neither will be spring supports, which could be introduced both with respect to the solid and entropy displacement, included. It appears that a consideration of such more general boundary conditions would not make it possible to carry out the proof, discussed in the last section of this paper, in the same manner as this is done for the more restricted class of boundary conditions.

The functions Φ, Σ, X and Θ are taken in the form

$$\Phi = U_i t_i \tag{2.11a}$$

$$\Sigma = S_i \theta_i \tag{2.11b}$$

$$X = T_i u_i \tag{2.11c}$$

$$\Theta = \Theta_i s_i \tag{2.11d}$$

where U_i, S_i, T_i and Θ_i are given functions on A_u, A_s, A_t and A_θ, respectively.

The first field equation of the set (2.1) is the static stress equation of equilibrium. The second equation might be called the thermal dis-equilibrium equation. The third equation expresses Hooke's law, relating linearly elastic isothermal stresses and strains. The fourth equation might be called the thermal stress-strain relation and it is a consequence of the principle of conservation of energy. The fifth equation expresses Fourier's law of heat conduction.

3. A general variational procedure. It is readily shown that the field equations (2.1) and the boundary conditions (2.2) may be derived as the Euler differential equations and the natural boundary conditions of the variational problem

$$\delta I = 0 \tag{3.1}$$

where

$$I = \int_v (\gamma_{ij}\tau_{ij} - \gamma\theta - s_i g_i - \psi - W - D)\, dV - \int_{A_u} (t_i u_i - \Phi)\, dA + \int_{A_s} (s_i \theta_i - \Sigma)\, dA - \int_{A_t} X\, dA + \int_{A_\theta} \Theta\, dA \tag{3.2}$$

and where the functions in each product have to be varied independently from each other. This general variational procedure is the extension of Reissner's principle for stresses and displacements of isothermal elasticity to the more general phenomena of thermoelasticity and heat conduction.

4. Variational principle for displacements. In this restricted principle the elastic isothermal stress-strain relation (2.1c) (Hooke's law) are considered as defining the isothermal stress, the thermal stress-strain relation (2.1d) are considered as defining the temperature, and the thermal disequilibrium equations (2.1e) are considered as defining the thermal disequilibrium force. Thus, the variation of these three dynamic quantities will be dependent upon the variation of the two displacements. On the bounding surface A the displacement variation is restricted, such that $\delta u_i = 0$ on A_u and $\delta s_i = 0$ on A_s .

The three stress-strain type relations mentioned above are inverted and written with the aid of two functions U and G as

$$\tau_{ij} = U_{,\gamma_{ij}} \tag{4.1a}$$

$$\theta = -U_{,\gamma} \tag{4.1b}$$

$$g_i = G_{,s_i} \tag{4.1c}$$

The functional I_u which yields the first two equations (2.1a), (2.1b) and the dynamic boundary conditions as a result of the variational procedure is

$$I_u = \int_v (U - G - \psi)\, dV - \int_{A_t} T_i u_i\, dA + \int_{A_\theta} \Theta_i s_i\, dA \tag{4.2}$$

It is seen that this procedure embodies an extension, to thermo-elasticity and heat conduction, of the variational principle for displacements due to Green.

5. Variational principle for stresses. Now the first two equations (2.1a), (2.1b) of the original set of five equations (2.1) are assumed to be satisfied, and, in addition, $\delta t_i = 0$ on A_t and $\delta\theta = 0$ on A_θ . The remaining three field equations and two boundary conditions (2.2a), (2.2b) may be derived variationally, if one sets

$$\delta I_\tau = 0 \tag{5.1}$$

where

$$I_r = \int_v (-W - D - \psi + u_i \psi_{,u_i})\, dV + \int_{A_u} U_i t_i \, dA - \int_{A_s} S_i \theta_i \, dA \qquad (5.2)$$

This procedure represents an extension of Castigliano's principle for stresses, in the formulation of Reissner [1], to thermoelasticity and heat conduction.

It is also noted, that Biot's [3] variational procedure, which yields the elastic equilibrium equations and the heat conduction equation as the Euler equations, is a mixed one, in the sense that the former is one of the equations obtained in applying the extended Green procedure, while the latter belongs to the equations obtained using the extended Castigliano's procedure.

6. Comparison of variational principles. Following Reissner's presentation and proof in isothermal elasticity, a comparison can be made between the values of I for functions τ_{ij}, g_i, etc. which are not solutions of $\delta I = 0$ and for functions τ_{ij}, etc. which are determined from $\delta I = 0$. If both W and D are positive definite quadratic forms, the conclusion is reached that in the extended variational theorem for displacements (solid and entropy) one is concerned with a minimum problem, while in the extended variational theorem for stresses one is concerned with a maximum problem. In contrast to this, Reissner's general variational theorem, as extended to thermoelasticity and heat conduction, is only a stationary-value problem.

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THE SOLUTION OF THE HEAT EQUATION SUBJECT TO THE SPECIFICATION OF ENERGY*

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1. Introduction. The purpose of this paper is to show that if the total heat energy of a certain part of a heat conductor is specified in advance as a function of time, the initial temperature of the conductor is known, and in the case of a finite conductor, the temperature behavior at one of the ends is specified in advance, then there exists a unique temperature distribution in the conductor which produces the specified total

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