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ON STRESSES AND DEFORMATIONS IN TOROIDAL SHELLS OF CIRCULAR CROSS SECTION WHICH ARE ACTED UPON BY UNIFORM NORMAL PRESSURE.*

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1. Introduction. We are concerned in what follows with the problem of toroidal shells of circular cross section which are acted upon by uniform normal pressure. It is well known that a solution of this problem by means of linear membrane theory, while leading to reasonable stress distributions, is associated with certain important inconsistencies insofar as the deformations of the shell are concerned [1]. It is also well known that a removal of these inconsistencies should be possible by an application of the linear theory of bending of toroidal shells [1]. However, for very thin shells it is more reasonable to expect that a removal of the deformational inconsistencies of linear membrane theory may be accomplished without considering wall bending action; that is, through the step from linear membrane theory to non-linear membrane theory. The first statement of this observation as well as an analysis of the non-linear membrane problem has been given by P. Jordan [2].

Our present object is a more general approach to the problem through a system of differential equations which contain both the equations of the linear bending theory and of the non-linear membrane theory as limiting cases, and which remains applicable in the transition region when both linear bending and non-linear membrane action have to be considered simultaneously. Specifically, we wish to determine the ranges of values of suitable non-dimensional parameters for which linear bending theory and non-linear membrane theory are appropriate, and also the ranges of these parameters for which the problem belongs to the transition region between the two limiting forms of the theory.

Derivation of the differential equations which govern the problem in all three ranges is accomplished through appropriate specialization of a general system of differential equations for finite symmetrical deflections of shells of revolution which has previously been given by the author [3].

2. Differential equations for finite symmetrical deflections of shells of revolution. Defining stress resultants, stress couples, and displacements in accordance with Fig. 1 and limiting attention to small strains, isotropy, and to vanishing transverse shear and normal strain, we have the following system of equilibrium equations and stress displacement relations for elements of the shell with middle surface equation $r = r(\xi)$, $z = z(\xi)$,

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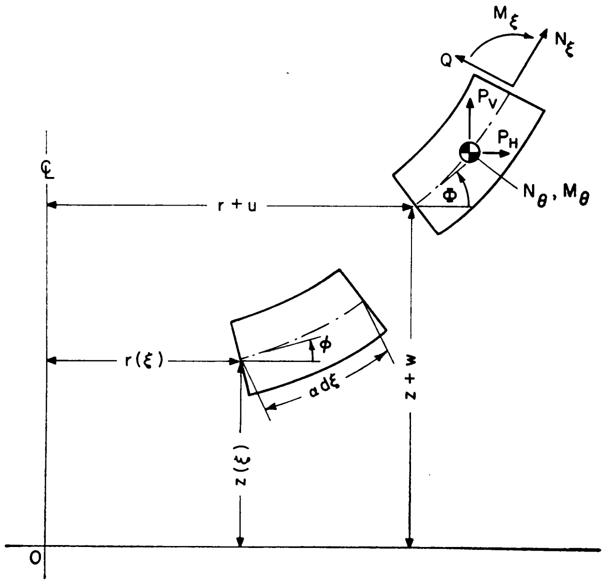

FIG. 1.

$$[r(N_\xi \sin \Phi + Q \cos \Phi)]' + r\alpha p_V = 0, \tag{1}$$

$$[r(N_\xi \cos \Phi - Q \sin \Phi)]' - \alpha N_\theta + r\alpha p_H = 0, \tag{2}$$

$$[rM_\xi]' - \alpha M_\theta \cos \Phi - r\alpha Q = 0, \tag{3}$$

$$\epsilon_\xi = \frac{u' - \alpha(\cos \Phi - \cos \phi)}{\alpha \cos \Phi} = \frac{N_\xi - \nu N_\theta}{C}, \tag{4}$$

$$\epsilon_\theta = u/r = (N_\theta - \nu N_\xi)/C, \tag{5}$$

$$M_\xi = D(\kappa_\xi + \nu\kappa_\theta) = -D\left(\frac{\Phi' - \phi'}{\alpha} + \nu\,\frac{\sin \Phi - \sin \phi}{r}\right) \tag{6}$$

$$M_\theta = D(\kappa_\theta + \nu\kappa_\xi) = -D\left(\frac{\sin \Phi - \sin \phi}{r} + \nu\,\frac{\Phi' - \phi'}{\alpha}\right) \tag{7}$$

In these equations, primes indicate differentiation with respect to ξ, $\alpha^2 = (r')^2 + (z')^2$, the stiffness factors C and D are of the form $C = Eh$, $D = Eh^3/12(1 - \nu^2)$, and κ_ξ, κ_θ are the curvatures.

To reduce the system (1) to (7) to a system of two simultaneous second order equations for the angular displacement variable Φ and for a stress function Ψ_H, we write

$$r(N_\xi \sin \Phi + Q \cos \Phi) = \Psi_V, \tag{8}$$

$$r(N_\xi \cos \Phi - Q \sin \Phi) = \Psi_H, \tag{9}$$

where, in view of Eq. (1)

$$\Psi_V = -\int r\alpha p_V \, d\xi. \tag{10}$$

and then have as expressions for stress resultants

$$rN_\xi = \Psi_V \sin \Phi + \Psi_H \cos \Phi, \tag{11}$$

$$rQ = \Psi_V \cos \Phi - \Psi_H \sin \Phi, \tag{12}$$

$$\alpha N_\theta = \Psi_H' + \alpha r p_H \,. \tag{13}$$

The first of the two simultaneous equations for Φ and Ψ_H follows by introducing Eqs. (6), (7) and (12) into the moment equilibrium equation (3), in the form

$$(\Phi - \phi)'' + \frac{(rD/\alpha)'}{(rD/\alpha)}(\Phi - \phi)' + \frac{\nu\alpha\phi'}{r}(\cos\Phi - \cos\phi)$$
$$- \left(\frac{\alpha^2}{r^2}\cos\Phi - \nu\frac{\alpha D'}{rD}\right)(\sin\Phi - \sin\phi) = \frac{\alpha^2}{rD}(\Psi_H \sin\Phi - \Psi_V \cos\Phi). \tag{I}$$

The second of the two simultaneous equations follows by introducing the stress displacement relations (4) and (5), with N_ξ and N_θ from (11) and (13), into a compatibility equation of the form

$$(r\epsilon_\theta)' - \alpha\epsilon_\xi \cos\Phi = \alpha(\cos\Phi - \cos\phi). \tag{14}$$

The resulting differential equation is

$$\Psi_H'' + \frac{(r/C\alpha)'}{(r/C\alpha)}\Psi_H' + \nu\frac{\alpha\Phi'}{r}\Psi_H \sin\Phi - \left(\frac{\alpha^2}{r^2}\cos\Phi - \nu\frac{\alpha C'}{rC}\right)\Psi_H \cos\Phi$$
$$= \frac{\alpha^2 C}{r}(\cos\Phi - \cos\phi) + \left(\frac{\alpha^2}{r^2}\cos\Phi - \nu\frac{\alpha C'}{rC}\right)\Psi_V \sin\Phi + \nu\frac{\alpha}{r}(\Psi_V \sin\Phi)'$$
$$- \left(2\alpha^2\cos\phi + \nu\alpha^2\cos\Phi - r\alpha\frac{C'}{C}\right)p_H - \alpha r p_H'. \tag{II}$$

Having expressions for stress resultants, stress couples and radial displacement component u in terms of Φ and Ψ_H it remains to express the axial displacement component w in analogous fashion. This is done by means of the relation

$$w' = (1 + \epsilon_\xi)\alpha\sin\Phi - \alpha\sin\phi. \tag{15}$$

Equations (1) to (15), together with (I) and (II) coincide, except for notation, with the contents of Sects. 2 and 3 of [3].

3. Differential equations for small finite deflections. We now write

$$\Phi = \phi + \beta \tag{16}$$

and retain the leading terms in expansions in powers of β, as follows

$$M_\xi = -D\left(\frac{\beta'}{\alpha} + \nu\frac{r'\beta}{r\alpha}\right), \qquad M_\theta = -D\left(\frac{r'\beta}{r\alpha} + \nu\frac{\beta'}{\alpha}\right). \tag{17}$$

$$rN_\xi = \Psi_V \sin\phi + \Psi_H \cos\phi + (\Psi_V \cos\phi - \Psi_H \sin\phi)\beta, \tag{18}$$

$$rQ = \Psi_V \cos\phi - \Psi_H \sin\phi - (\Psi_V \sin\phi + \Psi_H \cos\phi)\beta. \tag{19}$$

Equation (13) for N_θ remains as before, as does Eq. (5) for u. Equation (15) for w becomes

$$w' = \epsilon_\xi\alpha\sin\phi + \beta\alpha\cos\phi - \tfrac{1}{2}\alpha\beta^2\sin\phi, \tag{20}$$

and the two differential equations (I) and (II) are reduced to

$$\beta'' + \frac{(rD/\alpha)'}{rD/\alpha}\,\beta' - \left(\frac{r'^2}{r^2} - \nu\,\frac{(r'D/\alpha)'}{(rD/\alpha)}\right)\beta$$

$$= \frac{\alpha^2}{rD}[\Psi_H \sin\phi - \Psi_V \cos\phi + (\Psi_H \cos\phi + \Psi_V \sin\phi)\beta], \qquad \text{(III)}$$

$$\Psi_H'' + \frac{(r/C\alpha)'}{r/C\alpha}\,\Psi_H' - \left(\frac{r'^2}{r^2} + \nu\,\frac{(r'/C\alpha)'}{(r/C\alpha)}\right)\Psi_H$$

$$= -\frac{\alpha^2 C}{r}\,(\beta\sin\phi + \tfrac{1}{2}\beta^2 \cos\phi) + \left(\frac{\alpha^2\cos\phi}{r^2} - \nu\,\frac{\alpha C'}{rC}\right)\Psi_V \sin\phi + \nu\,\frac{\alpha}{r}\,(\Psi_V \sin\phi)'$$

$$- \left((2+\nu)\alpha^2\cos\phi - r\alpha\,\frac{C'}{C}\right)p_H - \alpha r p_H'. \qquad \text{(IV)}$$

Equations (III) and (IV) are equivalent to Eq. (III) and (IV) in [3], except that here we have omitted certain non-linear terms which are of minor significance.

4. Linear membrane theory. We indicate the values of quantities obtained from linear membrane theory by subscripts LM and define linear membrane theory through the relations

$$D = 0, \qquad (21)$$

$$Q_{LM} \equiv \Psi_{VLM} \cos\phi - \Psi_{HLM} \sin\phi = 0. \qquad (22)$$

Therewith, on the basis of (18) and (13),

$$N_{\xi LM} = \frac{\Psi_{HLM}}{r\cos\phi}, \qquad N_{\theta LM} = \frac{\Psi_{HLM}'}{\alpha} + r p_{HLM}\,. \qquad (23)$$

The quantities Ψ_{HLM} and Ψ_{VLM} satisfy the reduced version of Eq. (III) but do not satisfy the linearized version of Eq. (IV) with β_{LM} small. In fact, this latter equation requires in general that $\beta_{LM} = \infty$ whenever $\sin\phi = 0$.

5. Supplementation of linear membrane theory. We now remove the restrictive assumption of vanishing D and in addition to this consider supplementary stress resultants Q_S, $N_{\xi S}$, $N_{\theta S}$ and supplementary stress functions Ψ_{HS}, Ψ_{VS} such that

$$\Psi_H = \Psi_{HLM} + \Psi_{HS}\,, \qquad \Psi_V = \Psi_{VLM} + \Psi_{VS}\,. \qquad (24)$$

We further write

$$rN_{\xi S} = \Psi_{VS}\sin\phi + \Psi_{HS}\cos\phi, \qquad rQ_S = \Psi_{VS}\cos\phi - \Psi_{HS}\sin\phi \qquad (25)$$

and have then

$$N_\xi = N_{\xi LM} + N_{\xi S} + \beta Q_S\,, \qquad (26)$$

$$Q = Q_S - \beta N_{\xi LM} - \beta N_{\xi S}\,. \qquad (27)$$

Equations (26) and (27) are complemented by

$$N_\theta = N_{\theta LM} + N_{\theta S} \qquad (28)$$

where $N_{\theta LM}$ is given by (23) and

$$\alpha N_{\theta S} = \Psi_{HS}' + r\alpha p_{HS} \qquad (29)$$

It remains to obtain the differential equations for β and Ψ_{HS}, which follow from Equations (III) and (IV). Introduction of (24) together with (22), into Eq. (III) gives as the first of the two simultaneous equations for the determination of the supplementary solutions

$$\beta'' + \frac{(rD/\alpha)'}{(rD/\alpha)}\beta' - \left(\frac{r'^2}{r^2} - \nu\frac{(r'D/\alpha)'}{(rD/\alpha)}\right)\beta$$
$$= \frac{\alpha^2}{rD}(\Psi_{HS}\sin\phi - \Psi_{VS}\cos\phi) + \frac{\alpha^2 N_{\xi LM}}{D}\beta + \frac{\alpha^2}{rD}(\Psi_{HS}\cos\phi + \Psi_{VS}\sin\phi)\beta. \tag{V}$$

It is possible to obtain the second of the two simultaneous equations in a similar manner from Eq. (IV). A somewhat similar form of the results is obtained by returning to the compatibility equation (14) in the approximate form

$$(r\epsilon_\theta)' - \alpha\epsilon_\xi\cos\phi = -\alpha(\beta\sin\phi + \tfrac{1}{2}\beta^2\cos\phi) \tag{30}$$

and by neglecting the term βQ_S in Equation (26). In this way we obtain

$$\Psi''_{HS} + \frac{(r/C\alpha)'}{(r/C\alpha)}\Psi'_{HS} - \left(\frac{r'^2}{r^2} + \nu\frac{(r'/C\alpha)}{(r/C\alpha)}\right)\Phi_{HS}$$
$$= -\frac{\alpha^2 C}{r}(\beta\sin\phi + \tfrac{1}{2}\beta^2\cos\phi) + \left(\frac{\alpha^2\cos\phi}{r^2} - \nu\frac{\alpha C'}{rC}\right)\Psi_{VS}\sin\phi$$
$$+ \nu\frac{\alpha}{r}(\Psi_{VS}\sin\phi)' - \left((2+\nu)\alpha^2\cos\phi - r\alpha\frac{C'}{C}\right)p_{HS} - \alpha r p'_{HS}$$
$$- \frac{\alpha C}{r}\left[\left(r\frac{N_{\theta LM} - \nu N_{\xi LM}}{C}\right)' - \alpha\frac{N_{\xi LM} - \nu N_{\theta LM}}{C}\cos\phi\right] \tag{VI}$$

For many applications, in particular for the application to the problem of the toroidal shell with circular cross section and uniform internal pressure, Equation (V) and (VI) may be further simplified by omission of a number of terms, such that

$$\beta'' + \frac{(rD/\alpha)'}{rD/\alpha}\beta' - \left(\frac{r'^2}{r^2} - \nu\frac{(r'D/\alpha)'}{(rD/\alpha)}\right)\beta$$
$$= \frac{\alpha^2}{rD}(\Psi_{HS}\sin\phi - \Psi_{VS}\cos\phi) + \frac{\alpha^2 N_{\xi LM}}{D}\beta, \tag{VII}$$

$$\Psi''_{HS} + \frac{(r/C\alpha)'}{(r/C\alpha)}\Psi'_{HS} - \left(\frac{r'^2}{r^2} + \nu\frac{(r'/C\alpha)'}{(r/C\alpha)}\right)\Psi_{HS}$$
$$= -\frac{\alpha^2 C}{r}\beta\sin\phi - \frac{\alpha C}{r}\left[\left(r\frac{N_{\theta LM} - \nu N_{\xi LM}}{C}\right)' - \alpha\frac{N_{\xi LM} - \nu N_{\theta LM}}{C}\cos\phi\right]. \tag{VIII}$$

In other problems, such as in the problem of the flat plate subjected to lateral loads, omission of the non-linear portions involving Ψ_{HS} and β would not be admissible. We note that, consistent with the neglect of these terms in Eqs. (V) and (VI), the β^2-term in Eq. (20) may likewise be neglected.

6. Load terms with separation of effect of uniform normal pressure. We assume as expression for the load intensity components p_V and p_H in Eqs. (1), (2), (10), (11) and (II)

$$p_H = (1 + \epsilon_\xi)p\sin\Phi + p_{HA}, \tag{31}$$

$$p_V = -(1 + \epsilon_\xi)p \cos \Phi + p_{VA} , \tag{32}$$

where p is a uniform normal pressure and p_{HA} and p_{VA} are additional load intensity components such as inertia force intensities for rotating and or axially accelerating shells which are assumed to be unaffected by the deformation of the shell. We further write

$$p_{HLM} = p \sin \phi, \qquad p_{VLM} = -p \cos \phi, \tag{33}$$

and associate our linear membrane theory solution with this particular load distribution.

We then have from (10)

$$\Psi_V = p \int r\alpha(1 + \epsilon_\xi) \cos \Phi \, d\xi + \Psi_{VA} . \tag{34}$$

The integral for the normal pressure load p in (34) may be evaluated explicitly, if use is made of the compatibility equation (14) and of the assumption of small strain. We have, in view of (14)

$$\int r\alpha(1 + \epsilon_\xi) \cos \Phi \, d\xi = \int r[\alpha \cos \phi + (r\epsilon_\theta)'] \, d\xi$$

$$= \int r[r(1 + \epsilon_\theta)]' \, d\xi = (1 + \epsilon_\theta)r^2 - \int (1 + \epsilon_\theta)rr' \, d\xi. \tag{35}$$

Neglecting ϵ_θ in comparison with unity we have from (34)

$$\Psi_V = \tfrac{1}{2}pr^2 + \text{const.} + \Psi_{VA} . \tag{36}$$

We shall assume that the undeformed shell has tangent planes perpendicular to the shell axis for just one value $r = a$ and allocate part of the constant in (36) to Ψ_{VLM} such that $N_{\xi LM}$ and $N_{\theta LM}$ remain finite for $r = a$. This means that we have

$$\Psi_V = \Psi_{VLM} + \Psi_{VS} , \tag{37}$$

where

$$\Psi_{VLM} = \tfrac{1}{2}p(r^2 - a^2), \qquad \Psi_{VS} = k + \Psi_{VA}. \tag{38}$$

We further write instead of (31)

$$p_H = p_{HLM} + p_{HS} , \tag{39}$$

where p_{HLM} is given by (33) and where

$$p_{HS} = p\beta \cos \phi + p_{HA} , \tag{40}$$

7. Toroidal shell with uniform circular cross section subjected to uniform normal pressure. We take the equation of the middle surface of the shell in the form

$$r = a + b \sin \xi, \qquad z = -b \cos \xi \tag{41}$$

and have then

$$r' = b \cos \xi, \qquad z' = b \sin \xi, \qquad \alpha = b, \qquad \phi = \xi. \tag{42}$$

We assume that the stiffness factors C and D are independent of ξ and that $p_{VA} = p_{HA} = 0$. We set as abbreviations

$$b/a = \lambda, \qquad pb^3/D = \rho \tag{43}$$

and have then, from (38)

$$\Psi_{VLM} = pba(\sin \xi + \tfrac{1}{2}\lambda \sin^2 \xi) \tag{44}$$

and from (22), (23) and (33) the well known known formulas

$$N_{\xi LM} = pb \frac{1 + \frac{1}{2}\lambda \sin \xi}{1 + \lambda \sin \xi}, \qquad N_{\theta LM} = \tfrac{1}{2}pb \tag{45}$$

From (25) and (38) follows that

$$N_{\xi S} = \frac{k \sin \xi + \Psi_{HS} \cos \xi}{a(1 + \lambda \sin \xi)}, \qquad Q_S = \frac{k \cos \xi - \Psi_{HS} \sin \xi}{a(1 + \lambda \sin \xi)} \tag{46}$$

with N_ξ and Q given by (26) and (27).

From Equations (29) follows that

$$bN_{\theta S} = \Psi'_{HS} + pab(1 + \lambda \sin \xi)\beta \cos \xi \tag{47}$$

where the term with β will turn out to be small of higher order.

Expressions for stress couples follow from (17) in the form

$$M_\xi = -D\left(\frac{\beta'}{b} + \frac{\nu\lambda \cos \xi}{1 + \lambda \sin \xi} \frac{\beta}{b}\right), \qquad M_\theta = -D\left(\frac{\lambda \cos \xi}{1 + \lambda \sin \xi} \frac{\beta}{b} + \nu \frac{\beta'}{b}\right) \tag{48}$$

and Eq. (20) for the axial displacement w assumes the form

$$\frac{w'}{b} = \frac{N_\xi - \nu N_\theta}{C} \sin \xi + \beta \cos \xi \tag{49}$$

The differential equations (VII) and (VIII) for the determination of β and Ψ_{HS} now read as follows

$$\beta'' + \frac{\lambda \cos \xi}{1 + \lambda \sin \xi} \beta' - \left(\frac{\lambda^2 \cos^2 \xi}{(1 + \lambda \sin \xi)^2} + \frac{\nu\lambda \sin \xi}{1 + \lambda \sin \xi}\right)\beta$$
$$= \frac{b^2}{aD} \left(\frac{\sin \xi}{1 + \lambda \sin \xi} \Psi_{HS} - \frac{k \cos \xi}{1 + \lambda \sin \xi}\right) + \rho \frac{1 + \frac{1}{2}\lambda \sin \xi}{1 + \lambda \sin \xi} \beta, \tag{50}$$

$$\Psi''_{HS} + \frac{\lambda \cos \xi}{1 + \lambda \sin \xi} \Psi'_{HS} - \left(\frac{\lambda^2 \cos^2 \xi}{(1 + \lambda \sin \xi)^2} - \frac{\nu\lambda \sin \xi}{1 + \lambda \sin \xi}\right)\Psi_{HS}$$
$$= -\frac{b^2 C}{a} \frac{\sin \xi}{1 + \lambda \sin \xi} \beta + \frac{pb^3}{2a} \frac{\cos \xi}{(1 + \lambda \sin \xi)^2}. \tag{51}$$

Equations (50) and (51) are to be solved subject to appropriate boundary conditions. Four of such boundary conditions are the symmetry conditions of vanishing β and Q for $\xi = \pm\frac{1}{2}\pi$. Inspection of Eq. (27) and (46) indicates that the four symmetry conditions may be written in the form

$$\xi = \pm\tfrac{1}{2}\pi: \qquad \beta = 0, \qquad \Psi_{HS} = 0. \tag{52}$$

Two additional symmetry conditions are the conditions of vanishing axial displacement w for $\xi = \pm\pi/2$. Insofar as the solution of (50) and (51) is concerned, these two conditions may be replaced by the single condition $\int_{-\pi/2}^{\pi/2} w' \, d\xi = 0$ or, in view of (49), (47), (46) and (45), by

$$\int_{-\pi/2}^{\pi/2} \left[\frac{1}{C}\left(\frac{k \sin \xi + \Psi_{HS} \cos \xi}{a(1 + \lambda \sin \xi)} - \frac{\nu \Psi'_{HS}}{b} \right) \sin \xi + \beta \cos \xi \right] d\xi$$

$$= -\frac{pb}{C} \int_{-\pi/2}^{\pi/2} \frac{1 + \frac{1}{2}\lambda \sin \xi}{1 + \lambda \sin \xi} \sin \xi \, d\xi = \frac{pb}{C} \frac{\pi}{2\lambda} [(1 - \lambda^2)^{-1/2} - 1] \tag{53}$$

For the satisfaction of the five conditions (52) and (53) we have at our disposal four constants of integration in the solution of the two simultaneous second order equations (50) and (51), in addition to the constant value k of Ψ_{VS} .

Inspection of the system (50) to (53) indicates once more the necessity of the presence of the supplementary solution Ψ_{HS} to avoid β becoming infinite for $\xi = 0$ and $\xi = \pi$. It is further seen that both the linear bending theory formulation and the non-linear membrane theory formulation of the problem are included in (50) to (53), the former corresponding to the assumption $\rho = 0$, and the latter corresponding to the assumption $D = 0$ (and $\rho = \infty$) in Equation (50).

8. Non-dimensionalization of dependent variables. In order to see to what extent the problem is one without boundary layer phenomena, we consider the following non-dimensionalization

$$\beta = \beta_0 F(\xi), \qquad \Psi_{HS} = \ _0G(\xi), \qquad k = \Psi_0 K, \tag{54}$$

where

$$\Psi_0 = \frac{pb^3}{a}, \tag{55}$$

$$\frac{\Psi_0}{\beta_0} \frac{b^2}{aD} = \frac{\beta_0}{\Psi_0} \frac{b^2 C}{a} = \mu. \tag{56}$$

From (56) and (55) follows

$$\mu = \frac{b^2}{a}\left(\frac{C}{D}\right)^{1/2} = [12(1 - \nu^2)]^{1/2} \frac{b^2}{ah}, \tag{57}$$

$$\beta_0 = \frac{\Psi_0}{\mu} \frac{b^2}{aD} = \frac{pb^3\lambda^2}{\mu D} = \frac{\rho\lambda^2}{\mu}. \tag{58}$$

We note that $\lambda/\mu = (h/b)/[12(1 - \nu^2]^{1/2}$ is always small compared to unity.

The differential equations (50) and (51) are now

$$F'' + \frac{\lambda \cos \xi}{1 + \lambda \sin \xi} F' - \left(\frac{\lambda^2 \cos^2 \xi}{(1 + \lambda \sin \xi)^2} + \frac{\nu\lambda \sin \xi}{1 + \lambda \sin \xi} \right) F$$

$$= \frac{\mu \sin \xi}{1 + \lambda \sin \xi} G + \frac{\rho(1 + \frac{1}{2}\lambda \sin \xi)}{1 + \lambda \sin \xi} F - \frac{\mu K \cos \xi}{1 + \lambda \sin \xi}, \tag{59}$$

$$G'' + \frac{\lambda \cos \xi}{1 + \lambda \sin \xi} G' - \left(\frac{\lambda^2 \cos^2 \xi}{(1 + \lambda \sin \xi)^2} - \frac{\nu\lambda \sin \xi}{1 + \lambda \sin \xi} \right) G$$

$$= -\frac{\mu \sin \xi}{1 + \lambda \sin \xi} F + \frac{\frac{1}{2} \cos \xi}{(1 + \lambda \sin \xi)^2} \tag{60}$$

and the boundary conditions (52) and (53) are

$$\xi = \pm\pi/2: \qquad F = G = 0 \tag{61}$$

and

$$\int_{-\pi/2}^{\pi/2} \left[\left(\frac{K \sin \xi + G \cos \xi}{1 + \lambda \sin \xi} - \nu \frac{G'}{\lambda} \right) \sin \xi + \frac{\mu}{\lambda^2} F \cos \xi \right] d\xi = \frac{\pi}{2\lambda^3} [(1 - \lambda^2)^{-1/2} - 1] \tag{62}$$

Expressions for the relevant stress resultants and stress couples are, except for quantities which are small of higher order,

$$N_\xi = pb\left(\frac{1 + \frac{1}{2}\lambda \sin \xi}{1 + \lambda \sin \xi} + \lambda^2 \frac{G \cos \xi + K \sin \xi}{1 + \lambda \sin \xi} \right) \tag{63}$$

$$N_\theta = pb(\tfrac{1}{2} + \lambda G') \tag{64}$$

$$M_\xi = -\frac{pbh\lambda}{[12(1 - \nu^2)]^{1/2}} \left(F' + \frac{\nu\lambda \cos \xi}{1 + \lambda \sin \xi} F \right) \tag{65}$$

In order that the solution of (59) to (62) does not exhibit boundary layer phenomena we must have that μ as well as ρ are at most of order unity. This implies that the parameter λ must be small compared to unity, and this in turn implies that bending stresses as well as corrections to the membrane stresses of linear theory are small compared to the primary membrane stresses $\sigma_{\xi LM} = pb/h$ and $\sigma_{\theta LM} = pb/2h$. To the extent that such corrections are desired, they may be based on the abbreviated system

$$F'' = \mu G \sin \xi + \rho F, \tag{66}$$

$$G'' = -\mu F \sin \xi + \tfrac{1}{2} \cos \xi, \tag{67}$$

together with the boundary conditions (61), while the boundary condition (62) reduces to the condition that F be an odd function of ξ and G be an even function of ξ, with $K = 0$.

9. Non-dimensionalization of independent variable. It is implied by the structure of the differential equations (59) and (60) that when μ and or ρ are large compared to unity, the neighborhood of $\xi = 0$ is a region of relatively rapid changes of the dependent variables F and G, that is, an analysis of the boundary layer type is indicated. For the purpose of this analysis a new independent variable η is introduced in the form

$$\eta = \tau\xi \tag{68}$$

where $\tau \gg 1$ such that a narrow layer surrounding $\xi = 0$ becomes a layer surrounding $\eta = 0$, of width of order unity.

We further write in Equations (59) to (62)

$$F = F_0 f(\eta), \qquad G = G_0 g(\eta), \qquad K = G_0 c, \tag{69}$$

and, as we are concerned with the neighborhood of $\xi = 0$,

$$\sin \xi = \eta/\tau, \qquad \cos \xi = 1. \tag{70}$$

It remains to choose the parameters τ, F_0 and G_0. Two conditions for the determination of these parameters follow from Eq. (60) if we require that terms with F, G'' and 1/2 are of comparable magnitude, by setting

$$\tau^2 G_0 = \tau^{-1} \mu F_0 = 1. \tag{71}$$

A third condition follows from Eq. (59) in one of two ways. We may either require that

F'' and G terms are of comparable magnitude, which implies that bending action is significant, or we may require that the F and G terms are of comparable magnitude which implies that non-linear membrane action is significant.* Accordingly we set

$$\text{(i)} \quad \tau^2 F_0 = \tau^{-1}\mu G_0 \tag{72}$$

or

$$\text{(ii)} \quad \tau^{-1}\mu G_0 = \rho F_0 \,. \tag{73}$$

Solving (71) to (73) we have further

$$\text{(i)} \quad \tau = \mu^{1/3}, \qquad F_0 = \mu^{-2/3}, \qquad G_0 = \mu^{-2/3}, \tag{74}$$

or

$$\text{(ii)} \quad \tau = \mu^{1/2}\rho^{-1/4}, \qquad F_0 = \mu^{-1/2}\rho^{-1/4}, \qquad G_0 = \mu^{-1}\rho^{1/2}. \tag{75}$$

For both cases we have from (60) as one of the two simultaneous differential equations which are to be solved

$$g'' + \eta f = \tfrac{1}{2}, \tag{76}$$

and from Eq. (59)

$$\text{(i)} \qquad f'' - \eta g = \rho\mu^{-2/3} f - c \tag{77}$$

or

$$\text{(ii)} \qquad \mu\rho^{-3/2} f'' - \eta g = f - c. \tag{78}$$

Equations (76) and (77) or (78) are valid provided $\tau \gg 1$. In view of (74) and (75) we have then as *condition for the appropriateness of a boundary layer analysis,*

$$\operatorname{Max}\,(1,\, \rho^{1/6}) \ll \mu^{1/3}. \tag{79}$$

From Eq. (77) follows further that *linear bending theory is appropriate as long as*

$$\rho \ll \mu^{2/3}, \tag{80}$$

and from Eq. (78) follows that *non-linear membrane theory is appropriate as long as*

$$\mu^{2/3} \ll \rho. \tag{81}$$

In the transition region, say for $0.1\ \mu^{2/3} < \rho < 5\mu^{2/3}$, *both linear bending and non-linear membrane action have to be taken account of simultaneously.*

Boundary conditions for the solutions of the differential equations (76) and (77) or (78) follow from (61) and (62) in the form

$$\eta = \pm\infty: \qquad f = g = 0 \tag{82}$$

and, except for terms which are small of higher order,

$$\text{(i)} \quad \int_{-\mu^{1/3}}^{\mu^{1/3}} f(\eta)\, d\eta = \frac{\pi}{2\lambda}\left[(1 - \lambda^2)^{-1/2} - 1\right] \tag{83}$$

or

$$\text{(ii)} \quad \int_{-\mu^{1/2}\rho^{-1/4}}^{\mu^{1/2}\rho^{-1/4}} f(\eta)\, d\eta = \frac{\pi}{2\lambda}\left((1 - \lambda^2)^{-1/2} - 1\right). \tag{84}$$

Expressions for relevant stress resultants and stress couples follow from (63) to (65)

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*The analysis here is similar in nature to earlier work on the edge effect in bending of shells [4, 5].

in the form

$$\text{(i)}\quad N_\xi = pb\left(\frac{1+\frac{1}{2}\lambda\sin\xi}{1+\lambda\sin\xi} + \frac{\lambda^2}{\mu^{2/3}}\,g(\eta)\right), \tag{85}$$

$$\text{(ii)}\quad N_\xi = pb\left(\frac{1+\frac{1}{2}\lambda\sin\xi}{1+\lambda\sin\xi} + \frac{\lambda^2\rho^{1/2}}{\mu}\,g(\eta)\right); \tag{86}$$

$$\text{(i)}\quad N_\theta = pb\left(\tfrac{1}{2} + \frac{\lambda}{\mu^{1/3}}g'(\eta)\right) \tag{87}$$

$$\text{(ii)}\quad N_\theta = pb\left(\tfrac{1}{2} + \frac{\lambda\rho^{1/4}}{\mu^{1/2}}\,g'(\eta)\right); \tag{88}$$

$$\text{(i)}\quad M_\xi = -\frac{pbh\lambda}{[12(1-\nu^2)]^{1/2}}\left(\frac{1}{\mu^{1/3}}\,f'(\eta)\right), \tag{89}$$

$$\text{(ii)}\quad M_\xi = -\frac{pbh\lambda}{[12(1-\nu^2)]^{1/2}}\left(\frac{1}{\rho^{1/2}}\,f'(\eta)\right). \tag{90}$$

Equations (87) and (89) show that when consideration of wall bending is indicated, bending stresses $\sigma_{\xi B}$ and corrections to the values of the direct stresses $\sigma_{\theta D}$ as given by linear membrane theory are of the same order of magnitude and of order $\mu^{-1/3}$ relative to $\sigma_{\xi DLM}$. Corrections to $\sigma_{\xi D}$ are of order $\mu^{-2/3}$ relative to $\sigma_{\xi DLM}$.

In the range of parameter values for which non-linear membrane theory applies, it is found, in view of the fact that both $\rho^{1/2} \ll \mu$ and $\mu \ll \rho^{3/2}$, that $\sigma_{\xi B}$ as well as corrections to $\sigma_{\xi DLM}$ are small compared to the corrections to $\sigma_{\theta DLM}$ and that this correction in turn is small compared to $\sigma_{\theta DLM}$.

Accordingly, our analysis indicates that the distribution of stress as given by linear membrane theory is changed in a minor was only,* if account is taken of bending stresses and supplementary membrane stresses which are required to enforce continuity of displacements in the toroidal shell under uniform normal pressure, regardless of whether the problem is one of linear bending theory or of non-linear membrane theory, or one falling into the transition region between linear bending theory and non-linear membrane theory. Beyond establishing this particular fact, the equations of the present paper may be used for the analysis of pressurized shells of revolution under arbitrary symmetrical loads, under conditions where wall bending action and or the effect of pressurization are significant.

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*According to Jordan [2] these changes may be of the order of twenty per cent in the range for which non-linear membrane theory applies.