$$0 = 3y(r^2 - 5x^2)r^{-7} \frac{\partial x}{\partial \psi} + 3x(r^2 - 5y^2)r^{-7} \frac{\partial y}{\partial \psi} - 15xyzr^{-7} \frac{\partial z}{\partial \psi},$$

$$1 = -15xz^3/(4r^7) \frac{\partial x}{\partial \psi} - 15yz^3/(4r^7) \frac{\partial y}{\partial \psi} + (9z^2r^2 - 15z^4)/(4r^7) \frac{\partial z}{\partial \psi},$$

$$0 = 2xz^{-2} \frac{\partial x}{\partial \psi} - 2yz^{-2} \frac{\partial y}{\partial \psi} - 2(x^2 - y^2)z^{-3} \frac{\partial z}{\partial \psi},$$

which are obtained by differentiating $\Phi = 3xyr^{-5}$, $\psi = (3/4)z^3r^{-5}$, $\chi = (x^2 - y^2)z^{-2}$ with respect to ψ. The other sets result from differentiating with respect to Φ and χ. The Jacobian of Φ, ψ, χ with respect to x, y, z has the value:

$$J = 9r^{-12}[5x^2y^2 - r^2(x^2 + y^2)].$$

Now on $\Phi = \Phi_0$, the element of length ds is given by $ds^2 = E\,d\psi^2 + 2F\,d\psi\,d\chi + G\,d\chi^2$ where E, F, G have the usual meaning in terms of the derivatives with respect to ψ, and χ; and the element of surface area is $d\sigma = (EG - F^2)^{1/2}\,d\psi\,d\chi$. In view of the form of E, F, G this reduces to $J^{-1}\nabla\Phi\,d\psi\,d\chi$. The flux through $\Phi = \Phi_0$ is $\iint Q\cdot d\sigma = \iint (\nabla\psi \times \nabla\chi)\cdot J^{-1}\nabla\Phi\,d\psi\,d\chi$. But $\nabla\Phi\cdot\nabla\psi \times \nabla\chi = J$ so the flux is just $\iint d\psi\,d\chi$. If the area is bounded by the curves $\psi = \psi_1$, $\psi = \psi_2$ and $\chi = \chi_1$, $\chi = \chi_2$ the value obtained is $(\psi_2 - \psi_1)\cdot(\chi_2 - \chi_1)$, an obvious extension of the property which Stokes' stream function has.

Figure 1 shows two octants of the two stream surfaces with two members of the family of cones but only one of the closed surfaces. The streamlines in which these meet are indicated as heavy curves.

SYMMETRIC DUAL QUADRATIC PROGRAMS*

By RICHARD W. COTTLE (*Operations Research Center, University of California, Berkeley*)

1. Introduction. The duality theory of quadratic programming has been studied by Dennis [1] and principally by Dorn [2, 3, 4]. Wolfe [7] has specialized his results in nonlinear programming to the case of quadratic programming.

In this paper, two quadratic programs are presented which are dual, naturally symmetric, and related to a self-dual quadratic program. It is a consequence of the duality of these programs that if either has an optimal solution, then they share an optimal solution in common. Since duality, symmetry, and self-duality have each been studied by Dorn, some attention is given to the relation between the present work and his.

The programs to be considered are:

PRIMAL PROGRAM (P):

$$\begin{array}{lll} \text{Minimize} & F(x, y) = \tfrac{1}{2}y'\,Dy + \tfrac{1}{2}x'Cx + p'x & \text{(1a)} \\ \text{subject to} & Dy + Ax \geq -b & \text{(1b)} \\ \text{and} & x \geq 0; & \text{(1c)} \end{array}$$

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DUAL PROGRAM (P*):

$$\begin{array}{llr} \text{Maximize} & G(x, y) = -\tfrac{1}{2}v'Dv - \tfrac{1}{2}u'Cu - b'u & \text{(2a)} \\ \text{subject to} & -A'v + Cu \geq -p & \text{(2b)} \\ \text{and} & v \geq 0 & \text{(2c)} \end{array}$$

The entries of all matrices and vectors are real numbers. The symbols used in (1) and (2) have the following meanings: A is an $m \times n$ matrix; C, a symmetric positive semi-definite $n \times n$ matrix; D, a symmetric positive $m \times m$ matrix; b, an m-vector (column); y, v, m-vectors (column); p, an n-vector (column); x, u, n-vectors (column). The entries of A, C, D, b, and p are constants whereas those of x, y, u, and v are variables. An inequality between vectors means that the stated inequality holds between each of the corresponding components. The transpose of a vector or a matrix is denoted by a prime.

2. Notation and terminology. The primal program (P) will have the *constraint set*

$$P = \{(x, y) \mid Dy + Ax \geq -b, x \geq 0\},$$

and the dual program (P^*) will have the constraint set

$$P^* = \{(u, v) \mid -A'v + Cu \geq -p, v \geq 0\}.$$

An element of P or P^* is said to be a *feasible solution* of (P) or (P^*), respectively. A program is *infeasible* if its constraint set is empty. (The empty set will be denoted ϕ.)

A program is said to be *solvable* if its constraint set contains an element for which its *objective function* (F in (P) or G in (P^*)) attains the desired extremum. Such an element is called an *optimal solution* of the program and is said to *solve* it.

It will be shown that a relation of *duality* holds between (P) and (P^*) in the sense (see [7]) that:

(i) $\sup_{P^*} G \leq \inf_P F$;
(ii) the solvability of one problem implies that of the other, and the extremal values of F and G are equal;
(iii) if one problem is feasible while the other is not, then on its constraint set, the objective function of the feasible program is unbounded in the direction of extremization.

When D is a zero matrix, the dual programs of Dorn [2] appear. When both C and D are zero matrices, the familiar (von Neumann) symmetric dual linear programs result.

It is easy to see that if (P^*) is written as a minimization problem, then its dual is just (P) written as a maximization problem. Roughly speaking, then, the dual of the dual is the primal. It is this involutory property which Dorn [3] calls symmetry.

3. Duality. The first theorem establishes condition (i) in the definition of duality.

THEOREM 1. $\sup_{P^*} G \leq \inf_P F$.

Proof. Using the convention [7] that

$$\text{if } P^* = \phi, \quad \sup_{P^*} G = -\infty,$$
$$\text{if } P = \phi, \quad \inf_P F = +\infty,$$

it remains to prove the inequality under the assumption that both programs are feasible. Let $(u, v) \in P^*$, $(x, y) \in P$. Since $v \geq 0$ and $x \geq 0$, it follows that

$$-b'v - y'Dv \leq x'A'v \leq p'x + x'Cu.$$

Since C and D are symmetric and positive semi-definite, the inequalities [2, p. 156]

$$2y'Dv \leq y'Dy + v'Dv,$$

$$2x'Cu \leq x'Cx + u'Cu$$

may be applied to show that

$$-b'v - \tfrac{1}{2}y'Dy - \tfrac{1}{2}v'Dv \leq p'x + \tfrac{1}{2}x'Cx + \tfrac{1}{2}u'Cu$$

which yields

$$G(u, v) = -\tfrac{1}{2}v'\,Dv - \tfrac{1}{2}u'\,Cu - b'v \leq \tfrac{1}{2}y'\,Dy + \tfrac{1}{2}x'\,Cx + p'x = F(x, y).$$

This proves the theorem.

A solvable quadratic program is related to a certain linear program. By means of this correspondence, the duality theorem of linear programming may be employed. The proof follows that of Dorn in [2].

Lemma. If (x_0, y_0) is an optimal solution of (P), then it is an optimal solution of the linear program (L_0):

$$\text{Minimize} \quad f(x, y) = (y_0'\,D)y + (x_0'C)x + p'x \tag{3a}$$

$$\text{subject to} \quad Dy + Ax \geq -b \tag{3b}$$

$$\text{and} \quad x \geq 0 \tag{3c}$$

Proof. The constraint set P of program (P) is the same as that in (L_0). Suppose there exists $(x_1, y_1) \varepsilon P$ such that $f(x_1, y_1) - f(x_0, y_0) < 0$. This means

$$(y_0'D)(y_1 - y_0) + (x_0'C + p')(x_1 - x_0) < 0. \tag{4}$$

Let $0 < \lambda < 1$, and define

$$x^* = (1 - \lambda)x_0 + \lambda x_1 = x_0 + \lambda(x_1 - x_0),$$

$$y^* = (1 - \lambda)y_0 + \lambda y_1 = y_0 + \lambda(y_1 - y_0).$$

By the convexity of P, $(x^*, y^*) \varepsilon P$. Consider

$$F(x^*, y^*) - F(x_0, y_0) = \lambda[(y_0'D)(y_1 - y_0) + (x_0'C + p')(x_1 - x_0)] + (\lambda^2/2)[(y_1 - y_0)'D(y_1 - y_0) + (x_1 - x_0)'C(x_1 - x_0)]. \tag{5}$$

It follows from (4) that the right-hand side of (5) can be made negative by taking sufficiently small positive λ. This contradicts the assumption of the optimality of (x_0, y_0) in the program (P). Therefore (x_0, y_0) must be optimal for (L_0).

THEOREM 2. If (P) is solvable, then (P^*) is solvable, and the extremal values of F and G are equal.

Proof. Let (x_0, y_0) solve (P). By Theorem 1, if there exists $(u_0, v_0) \varepsilon P^*$ such that $G(u_0, v_0) = F(x_0, y_0)$, then (u_0, v_0) solves (P^*). By the Lemma, (x_0, y_0) is optimal for (L_0). The duality theorem of linear programming (cf. [1]) states that there exists a vector v_0 such that

$$-A'v_0 + Cx_0 \geq -p, \qquad v_0 \geq 0, \qquad v_0'D = y_0'D,$$

and

$$-b'v_0 = y_0'Dy_0 + x_0'Cx_0 + p'x_0. \tag{6}$$

Notice that $(x_0, v_0) \varepsilon P^*$. By the symmetry of D

$$v_0'Dv_0 = y_0'Dv_0 = v_0'Dy_0 = y_0'Dy_0.$$

Thus from (6)

$$\begin{aligned} F(x_0, y_0) &= \tfrac{1}{2}y_0'Dy_0 + \tfrac{1}{2}x_0'Cx_0 + p'x_0 \\ &= -\tfrac{1}{2}y_0'Dy_0 - \tfrac{1}{2}x_0'Cx_0 - b'v_0 \\ &= -\tfrac{1}{2}v_0'Dv_0 - \tfrac{1}{2}x_0'Cx_0 - b'v_0 \\ &= G(x_0, v_0), \end{aligned}$$

and the proof is complete.

The demonstration of Theorem 2 provides the important

Corollary 1. If (x_0, y_0) solves (P), there exists a vector v_0 such that (x_0, v_0) solves (P^*) and $G(x_0, v_0) = F(x_0, y_0)$. Moreover, $Dy_0 = Dv_0$.

This corollary and the symmetry discussed above yield

Corollary 2. If (u_0, v_0) solves (P^*), there exists a vector x_0 such that (x_0, v_0) solves (P) and $G(u_0, v_0) = F(x_0, v_0)$. Moreover, $Cu_0 = Cx_0$.

Condition (ii) for duality is now satisfied. A further observation can be made.

Corollary 3. Non-negativity restrictions may be imposed on all the variables in (P) and (P^*) without affecting the question of their solvability.

Condition (iii) is a consequence of

THEOREM 3. If (P) is feasible and (P^*) is infeasible, then $\inf_P F = -\infty$.

Proof. Let $(x, y) \varepsilon P$. The assumption $P^* = \phi$ means there exist no pair (u, v) satisfying

$$\begin{pmatrix} -A' & C \\ I & 0 \end{pmatrix} \begin{pmatrix} v \\ u \end{pmatrix} \geq \begin{pmatrix} -p \\ 0 \end{pmatrix}$$

It follows (cf. [6], p. 46) that there exist vectors $x^* \geq 0$ and $y^* \geq 0$ such that

$$\begin{pmatrix} -A & I \\ C & 0 \end{pmatrix} \begin{pmatrix} x^* \\ y^* \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$(-\mathrm{p}', 0) \begin{pmatrix} x^* \\ y^* \end{pmatrix} = 1.$$

In particular,

$$Iy^* = Ax^* \geq 0, \qquad Cx^* = 0, \qquad p'x^* = -1.$$

With $\lambda \geq 0$ it follows that $(x + \lambda x^*, y) \varepsilon P$. However,

$$F(x + \lambda x^*, y) = -\lambda + F(x, y).$$

Clearly $F(x + \lambda x^*, y) \to -\infty$ as $\lambda \to +\infty$.

Again the symmetric result holds.

Corollary 1. If (P^*) is feasible and (P) is infeasible, then $\sup_{P^*} G = +\infty$.

An immediate consequence of Theorems 1 and 3 and Corollary 1 is

Corollary 2. If either program (P) or (P^*) is feasible, its objective function is bounded in the direction of extremization if and only if the other problem is feasible.

The duality relation is now fully established.

THEOREM 4. (Joint solution). If (P) or (P^*) is solvable, there exist vectors x_0 and v_0 such that (x_0, v_0) is an optimal solution for (P) and (P^*).

Proof. It is no restriction to assume that (P) is solvable. Let (x_0, y_0) solve (P). By Corollary 1, there exists a vector v_0 such that (x_0, v_0) solves (P^*),

$$G(x_0, v_0) = F(x_0, y_0) \quad \text{and} \quad Dy_0 = Dv_0.$$

From this it follows that $(x_0, v_0) \in P$ and $G(x_0, v_0) = F(x_0, v_0)$. Hence (x_0, v_0) solves (P).

Remark. (Complementary slackness). Let (x_0, v_0) be a joint solution of (P) and (P^*). Define

$$s_0 = Dv_0 + Ax_0 + b, \qquad t_0 = -A'v_0 + Cx_0 + p.$$

It follows that

$$s_0'v_0 = 0 \quad \text{and} \quad t_0'x_0 = 0, \tag{7}$$

for by the hypothesis on (x_0, v_0),

$$s_0'v_0 = v_0'Dv_0 + x_0'A'v_0 + b'v_0 \geq 0, \tag{8a}$$

$$t_0'x_0 = -v_0'Ax_0 + x_0'Cx_0 + p'x_0 \geq 0, \tag{8b}$$

$$\tfrac{1}{2}v_0'Dv_0 + \tfrac{1}{2}x_0'Cx_0 + p'x_0 = -\tfrac{1}{2}v_0'Dv_0 - \tfrac{1}{2}x_0'Cx_0 - b'v_0. \tag{9}$$

From (9) it is clear that $s_0'v_0 + t_0'x_0 = 0$ which, by (8), can happen only if each of the summands is zero.

In linear programming, the feasibility of both the primal and dual programs implies the existence of optimal solutions for each of them. It is possible to make this claim for quadratic programming.

THEOREM 5. If (P) and (P^*) are feasible, then (P) and (P^*) are solvable.

Proof. Since P and P^* are nonempty, F is bounded below on P and G is bounded above on P^*. The result of Frank and Wolfe [5, App. i] may be modified to show that G must attain its supremum (over P^*) on P^*. The remainder of the proof is an application of Corollary 2 of Theorem 2.

4. Self-duality. The self-duality presented in [4] requires the notion of *equivalence*, particularly since the dual programs considered there are not of the same form. However, symmetric dual quadratic programs permit a *formal* type of self-duality which resembles that for linear programs. The linear program (L_1)

$$\begin{array}{ll} \text{Minimize} & q'z \\ \text{subject to} & A_1 z \geq -q \\ \text{and} & z \geq 0, \end{array}$$

where A_1 is skew symmetric is clearly self-dual. If the program is feasible, Min $q'z = 0$. Indeed, the dual of this program is formally the same program. The analogous statement for quadratic programming is

THEOREM 6. The quadratic program (P_1)

$$\begin{array}{ll} \text{Minimize} & \Phi(X, Y) = \tfrac{1}{2}Y'C_1Y + \tfrac{1}{2}X'C_1X' + q'X \\ \text{subject to} & C_1Y + A_1X \geq -q \\ \text{and} & X \geq 0, \end{array}$$

where A_1 is skew symmetric and C_1 is symmetric positive semi-definite, is self-dual. Moreover, if (P_1) is feasible, then Min $\Phi(X, Y) = 0$.

Proof. The dual of (P_1) is the program (P_1^*)

$$\begin{array}{ll} \text{Maximize} & \Psi(U, V) = -\tfrac{1}{2}V'C_1V - \tfrac{1}{2}U'C_1U - q'V \\ \text{subject to} & -A_1'V + C_1U \geq -q \\ \text{and} & V \geq 0. \end{array}$$

Since A_1 is skew symmetric, (P_1) is self-dual. Again by the skew symmetry of A_1, $X'A_1X = 0$ for all X. Let $(X, Y) \varepsilon P_1$. It follows that

$$Y'C_1X \geq -q'X.$$

Therefore

$$\Phi(X, Y) = \tfrac{1}{2}Y'C_1Y + \tfrac{1}{2}X'C_1X + q'X \geq (Y - X)'C_1(Y - X) \geq 0.$$

The program (P_1) is solvable since Φ is bounded below on P_1. Consequently (P_1^*) is also solvable. For all $(U, V) \varepsilon P_1^*$, $\Psi(U, V) \leq 0$. Now let (X_0, Y_0) be a joint solution for (P_1) and (P_1^*), then $0 \leq \Phi(X_0, Y_0) = \Psi(X_0, Y_0) \leq 0$. The conclusion follows.

A program of the form (L_1) may be obtained as a composite of the

PRIMAL PROGRAM (L):

$$\begin{array}{ll} \text{Minimize} & p'x \\ \text{subject to} & Ax \geq -b \\ \text{and} & x \geq 0 \end{array}$$

and the DUAL PROGRAM (L*):

$$\begin{array}{ll} \text{Maximize} & -b'v \\ \text{subject to} & -A'v \geq -p \\ \text{and} & v \geq 0, \end{array}$$

simply by taking

$$A_1 = \begin{bmatrix} 0 & A \\ -A' & 0 \end{bmatrix}, \quad q = \begin{bmatrix} b \\ p \end{bmatrix}, \quad z = \begin{bmatrix} v \\ x \end{bmatrix}.$$

In a similar manner, the composite of the programs (P) and (P^*) yields one of the form (P_1). The identifications are

$$A_1 = \begin{bmatrix} 0 & A \\ -A' & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} D & 0 \\ 0 & C \end{bmatrix}, \quad q = \begin{bmatrix} b \\ p \end{bmatrix}, \quad X = \begin{bmatrix} v \\ x \end{bmatrix}, \quad Y = \begin{bmatrix} y \\ u \end{bmatrix}$$

and $\Phi(X, Y) = F(x, y) - G(u, v)$.

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APPLICATION OF VARIATIONAL PRINCIPLES TO LIMIT ANALYSIS*

BY T. MURA AND S. L. LEE (*Civil Engineering Department, Northwestern University*)

1. Introduction. The use of variational principles to derive kinematically admissible velocity fields and statically admissible stress fields in the limit analysis of perfectly plastic structures [1-6] has been discussed by several investigators [7-13]. The purpose of this paper is to demonstrate by means of these principles that the safety factor, the kinematically admissible multiplier and the statically admissible multiplier for a body made of perfectly plastic material and subjected to given surface traction are actually extremum values of the same functional under different constraint conditions.

The kinematically admissible multiplier m^* is defined by the relation[1]

$$m^* = k \int_V (2\epsilon_{ij}^* \epsilon_{ij}^*)^{1/2}\, dV \Big/ \int_{S_T} T_i v_i^*\, dS, \tag{1}$$

where

$$\epsilon_{ij}^* = (v_{i,j}^* + v_{j,i}^*)/2 \quad \text{in} \quad V, \tag{2}$$

$$\delta_{ij} v_{i,j}^* = 0 \quad \text{in} \quad V, \tag{3}$$

$$v_i^* = 0 \quad \text{on} \quad S_V, \tag{4}$$

$$\int_{S_T} T_i v_i^*\, dS > 0 \tag{5}$$

In these equations, V denotes a closed domain bounded by a closed surface S, S_T a portion of S subjected to given surface traction T_i while on the remainder S_V the velocities are prescribed to vanish, ϵ_{ij}^* the strain rate field associated with the velocity field v_i^*, and k is the yield limit in simple shear. A velocity field is said to be kinematically admissible if it satisfies (3), (4) and (5).

The statically admissible multiplier is defined as follows: A stress field σ_{ij}^0 is said to be statically admissible if

$$\sigma_{ij,j}^0 = 0 \quad \text{in} \quad V, \tag{6}$$

$$\sigma_{ij}^0 n_j = m^0 T_i \quad \text{on} \quad S_T, \tag{7}$$

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[1]See [1], p. 247.