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APPLICATION OF VARIATIONAL PRINCIPLES TO LIMIT ANALYSIS*

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1. Introduction. The use of variational principles to derive kinematically admissible velocity fields and statically admissible stress fields in the limit analysis of perfectly plastic structures [1-6] has been discussed by several investigators [7-13]. The purpose of this paper is to demonstrate by means of these principles that the safety factor, the kinematically admissible multiplier and the statically admissible multiplier for a body made of perfectly plastic material and subjected to given surface traction are actually extremum values of the same functional under different constraint conditions.

The kinematically admissible multiplier m^* is defined by the relation[1]

$$m^* = k \int_V (2\epsilon_{ij}^*\epsilon_{ij}^*)^{1/2}\, dV \Big/ \int_{S_T} T_i v_i^*\, dS, \tag{1}$$

where

$$\epsilon_{ij}^* = (v_{i,j}^* + v_{j,i}^*)/2 \quad \text{in} \quad V, \tag{2}$$

$$\delta_{ij} v_{i,j}^* = 0 \quad \text{in} \quad V, \tag{3}$$

$$v_i^* = 0 \quad \text{on} \quad S_V, \tag{4}$$

$$\int_{S_T} T_i v_i^*\, dS > 0 \tag{5}$$

In these equations, V denotes a closed domain bounded by a closed surface S, S_T a portion of S subjected to given surface traction T_i while on the remainder S_V the velocities are prescribed to vanish, ϵ_{ij}^* the strain rate field associated with the velocity field v_i^*, and k is the yield limit in simple shear. A velocity field is said to be kinematically admissible if it satisfies (3), (4) and (5).

The statically admissible multiplier is defined as follows: A stress field σ_{ij}^0 is said to be statically admissible if

$$\sigma_{ij,j}^0 = 0 \quad \text{in} \quad V, \tag{6}$$

$$\sigma_{ij}^0 n_j = m^0 T_i \quad \text{on} \quad S_T, \tag{7}$$

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[1]See [1], p. 247.

$$f(s_{ij}^0) = \tfrac{1}{2}s_{ij}^0 s_{ij}^0 - k^2 \leq 0 \quad \text{in} \quad V, \tag{8}$$

where

$$s_{ij}^0 = \sigma_{ij}^0 - \delta_{ij}\sigma^0, \tag{9}$$

$$\sigma^0 = 1/3\sigma_{kk}^0. \tag{10}$$

The proportional constant m^0 defined by (7) is termed the statically admissible multiplier.

The ratio s of a generic surface traction at the instant of impending plastic flow to the given value of the surface traction T_i is called the safety factor. It is well known that

$$m^0 \leq s \leq m^* \tag{11}$$

2. Safety factor. Consider the minimum problem of the functional $F_1[v_i\, ,\, s_{ij}]$ with independent arguments v_i and s_{ij}. The existence of the minimum problem will be proven later.

Problem I. Minimize

$$F_1[v_i, s_{ij}] = \int_V s_{ij}\tfrac{1}{2}(v_{i,j} + v_{j,i})\, dV \tag{12}$$

with the constraint conditions

$$f(s_{ij}) = \tfrac{1}{2}s_{ij}s_{ij} - k^2 \leq 0 \quad \text{in} \quad V, \tag{13}$$

$$(s_{ij} - s'_{ij})\tfrac{1}{2}(v_{i,j} + v_{j,i}) \geq 0 \quad \text{in} \quad V, \tag{14}$$

for any s'_{ij} such that $f(s'_{ij}) \leq 0$,

$$\delta_{ij}v_{i,j} = 0 \quad \text{in} \quad V, \tag{15}$$

$$v_i = 0 \quad \text{on} \quad S_V, \tag{16}$$

$$\int_{S_T} T_i v_i\, ds = 1. \tag{17}$$

In order to determine the minimal conditions, the problem is transformed by employing point-function Lagrangian multipliers σ, R_i, and μ and a constant Lagrangian multiplier m as follows.

Problem II. Extremize

$$\begin{aligned} F_2[v_i, s_{ij}, \sigma, R_i, m, \mu, \varphi] = {} & \int_V s_{ij}\tfrac{1}{2}(v_{i,j} + v_{j,i})\, dV \\ & + \int_V \sigma\delta_{ij}v_{i,j}\, dV - \int_{S_V} R_i v_i\, dS - m\left(\int_{S_T} T_i v_i\, dS - 1\right) \\ & - \int_V \mu[f(s_{ij}) + \varphi^2]\, dV \end{aligned} \tag{18}$$

with the constraint condition

$$\mu \geq 0. \tag{19}$$

The point function φ is introduced on account of the inequality of condition (13) and, as will be shown later, constraint condition (19) is imposed in order to take into consideration constraint condition (14).

Taking the variation of F_2 leads to

$$\begin{aligned}\delta F_2 = {} & \int_V \delta s_{ij}\tfrac{1}{2}(v_{i,j} + v_{j,i})\, dV + \int_V s_{ij}\tfrac{1}{2}(\delta v_{i,j} + \delta v_{j,i})\, dV \\ & + \int_V \delta\sigma\,\delta_{ij}v_{i,j}\, dV + \int_V \sigma\,\delta_{ij}\,\delta v_{i,j}\, dV - \int_{S_V} \delta R_i v_i\, dS - \int_{S_V} R_i\,\delta v_i\, dS \\ & - \delta m\left(\int_{S_T} T_i v_i\, dS - 1\right) - m\int_{S_T} T_i\,\delta v_i\, dS - \int_V \delta\mu[f(s_{ij}) + \varphi^2]\, dV \\ & - \int_V \mu\frac{\partial f}{\partial s_{ij}}\,\delta s_{ij}\, dV - \int_V \mu 2\varphi\,\delta\varphi\, dV \end{aligned} \tag{20}$$

Integrating (20) by parts yields the natural conditions

$$\tfrac{1}{2}(v_{i,j} + v_{j,i}) = \mu\frac{\partial f}{\partial s_{ij}} \quad \text{in} \quad V, \tag{21}$$

$$\mu \geq 0,$$

$$(s_{ij} + \delta_{ij}\sigma)_{,j} = 0 \quad \text{in} \quad V, \tag{22}$$

$$(s_{ij} + \delta_{ij}\sigma)n_j = mT_i \quad \text{on} \quad S_T, \tag{23}$$

$$(s_{ij} + \delta_{ij}\sigma)n_j = R_i \quad \text{on} \quad S_V, \tag{24}$$

$$f(s_{ij}) + \varphi^2 = 0 \quad \text{in} \quad V, \tag{25}$$

$$\mu\varphi = 0 \quad \text{in} \quad V, \tag{26}$$

$$\delta_{ij}v_{i,j} = 0 \quad \text{in} \quad V, \tag{27}$$

$$v_i = 0 \quad \text{on} \quad S_V, \tag{28}$$

$$\int_{S_T} T_i v_i = 1. \tag{29}$$

Condition (21) is the plastic potential flow law, (22) to (24) are the equilibrium conditions, and (27) to (29) define a kinematically admissible velocity field. Conditions (25) and (26) define the admissible domain of the stress space, i.e.,

$$f(s_{ij}) = 0 \quad \text{if} \quad \mu > 0, \tag{30}$$

$$f(s_{ij}) \leq 0 \quad \text{if} \quad \mu = 0. \tag{31}$$

Obviously, (21) to (29) are the conditions for incipient plastic flow. Condition (24) should be understood as an equation for determining R_i, the reaction on the boundary, which is arbitrary. It should be mentioned here that condition (29) is no more restrictive than the requirement

$$\int_{S_T} T_i v_i\, dS > 0.$$

Setting the integral equal to unity only determines the scale of the otherwise arbitrary size of the velocity vector. It should also be observed that constraint condition (14) on

F_1 is satisfied by (21) and the fact that the yield surface, $f(s_{ij}) = 0$, is convex with respect to the origin of the vector space.

Integrating the functionals by parts in view of (21) to (29), it can be readily shown that the minimum value of F_1 and the extremum value of F_2 are equal to m. Therefore the safety factor can be defined as the minimum of F_1 or the extremum value of F_2, i.e.,

$$\text{Min. } F_1 = \text{Ext. } F_2 = m = s \tag{32}$$

To prove the existence of the minimum problem for F_1, it is sufficient to show that

$$\int_V s_{ij}^* \tfrac{1}{2}(v_{i,j}^* + v_{j,i}^*)\, dV \geq \int_V s_{ij} \tfrac{1}{2}(v_{i,j} + v_{j,i})\, dV \tag{33}$$

for an set of v_i^* and s_{ij}^* which satisfy the constraint conditions (13) to (17) and v_i and s_{ij} are the stationary functions. Since conditions (22) to (29) are applicable to v_i and s_{ij},

$$\begin{aligned}\int_V s_{ij}\tfrac{1}{2}(v_{i,j} + v_{j,i})\, dV &= \int_V \sigma_{ij} v_{i,j}\, dV = \int_{S_T} \sigma_{ij} v_i n_j\, dS \\ &= m\int_{S_T} T_i v_i\, dS = m\int_{S_T} T_i v_i^*\, dS = \int_S \sigma_{ij} v_i^* n_j\, dS \\ &= \int_V \sigma_{ij,j} v_i^*\, dV + \int_V \sigma_{ij} v_{i,j}^*\, dV = \int_V s_{ij}\tfrac{1}{2}(v_{i,j}^* + v_{j,i}^*)\, dV \end{aligned} \tag{34}$$

In view of (14),

$$(s_{ij}^* - s_{ij})\tfrac{1}{2}(v_{i,j}^* + v_{j,i}^*) \geq 0 \tag{35}$$

Thus, statement (33) is established and the existence of the minimum problem is verified.

3. Kinematically admissible multiplier. It can be shown that a kinematically admissible velocity field v_i^* and the associated stress field s_{ij}^* constructed from v_i^* by means of

$$s_{ij}^* = k\dot{\epsilon}_{ij}^*/(\tfrac{1}{2}\dot{\epsilon}_{mn}^*\dot{\epsilon}_{mn}^*)^{1/2}, \tag{36}$$

where $\dot{\epsilon}_{ij}^*$ is defined by (2), are admissible comparison functions of Problem I. As a matter of fact, substituting

$$v_i = v_i^* \Big/ \int_{S_T} T_i v_i^*\, dS, \tag{37}$$

$$s_{ij} = s_{ij}^* \tag{38}$$

in (12) yields

$$F_1 = m^*, \tag{39}$$

where m^* is as defined by (1). Since the safety factor is the minimum value of F_1, the following inequality holds:

$$m^* \geq s, \tag{40}$$

4. Statically admissible multiplier. In accordance with the maximal minimum principle of calculus of variations[2] another minimum problem may be obtained from

[2]See [14], p. 232.

Problem II for fixed values of the Lagrangian multipliers σ^0, R_i^0, m^0 and μ^0 with additional constraint conditions as follows.
Problem III. Minimize

$$F_3[v_i, s_{ij}] = \int_V s_{ij}\tfrac{1}{2}(v_{i,j} + v_{j,i})\,dV + \int_V \sigma^0\delta_{ij}v_{i,j}\,dV$$
$$- \int_{S_V} R_i^0 v_i\,dS - m^0\left(\int_{S_T} T_i v_i\,dS - 1\right) - \int_V \mu^0[f(s_{ij}) + \varphi^2]\,dV \qquad (41)$$

with constraint conditions

$$f(s_{ij}) + \varphi^2 = 0 \quad \text{in} \quad V, \qquad (42)$$
$$(s_{ij} - s'_{ij})\tfrac{1}{2}(v_{i,j} + v_{j,i}) \geq 0 \quad \text{in} \quad V, \qquad (43)$$

for any s'_{ij} such that $f(s'_{ij}) \leq 0$.
Taking the variation of F_3 with respect to v_i leads to

$$(s_{ij} + \delta_{ij}\sigma^0)_{,j} = 0 \quad \text{in} \quad V, \qquad (44)$$
$$(s_{ij} + \delta_{ij}\sigma^0)n_j = m^0 T_i \quad \text{on} \quad S_T, \qquad (45)$$
$$(s_{ij} + \delta_{ij}\sigma^0)n_j = R_i^0 \quad \text{on} \quad S_V. \qquad (46)$$

Integrating (41) by parts in view of (44) to (46) leads to

$$F_3 = m^0 - \int_V \mu^0[f(s_{ij}) + \varphi^2]\,dV \qquad (47)$$

which, in view of (42), simplifies finally into

$$\text{Min. } F_3 = m^0. \qquad (48)$$

In order to prove that Problem III is a minimum problem, it is sufficient to show that for any set of arbitrary comparison functions v_i^0 and s_{ij}^0,

$$F_3[v_i^0, s_{ij}^0] \geq F_3[v_i, s_{ij}], \qquad (49)$$

in which v_i and s_{ij} are the stationary functions. In view of (47),

$$F_3[v_i, s_{ij}] = F_3[v_i^0, s_{ij}]. \qquad (50)$$

Therefore, taking (42) and (43) into consideration,

$$F_3[v_i^0, s_{ij}^0] - F_3[v_i, s_{ij}] = F_3[v_i^0, s_{ij}^0] - F_3[v_i^0, s_{ij}]$$
$$= \int_V (s_{ij}^0 - s_{ij})\tfrac{1}{2}(v_{i,j}^0 + v_{j,i}^0)\,dV \geq 0 \qquad (51)$$

and (49) holds true.

Problem I may be made up from Problem III by superimposing, on top of (42) and (43), the following additional constraint conditions:

$$\delta_{ij}v_{i,j} = 0 \quad \text{in} \quad V, \qquad (52)$$
$$v_i = 0 \quad \text{on} \quad S_V, \qquad (53)$$
$$\int_{S_T} T_i v_i\,dS = 1. \qquad (54)$$

Since (52) to (54) are not natural conditions of Problem III, they raise the minimum value of the functional. Hence the minimum of F_1 cannot be less than the minimum value of F_3. In other words,

$$m^0 \leq s \tag{55}$$

Since conditions (42), (44) and (45) are identical to (6), (7) and (8), m^0 in (55) turns out to be a statically admissible multiplier.

5. Conclusion. From the foregoing discussions, the statically admissible multiplier m^0 can be defined as the minimum value of F_3 with the constraint conditions (42) and (43), or the value of F_3 calculated from a statically admissible stress field.

Since a kinematically admissible velocity field satisfies conditions (52) to (54) and $F_3 = F_1$ under these additional constraint conditions, the kinematically admissible multiplier m^* can be defined as the value of F_3 or F_1 calculated from a kinematically admissible velocity field and a stress field with constraint conditions (42) and (43).

The safety factor can be defined as the maximal minimum value of F_3 with respect to the Lagrangian multipliers σ^0, R_i^0, m^0 and μ^0, or the minimum value of F_3 or F_1 with the constraint conditions (42), (43), (52), (53) and (54). It can also be defined as the extremum value of F_2 with the constraint condition (19).

It should be noted that the foregoing discussions can be extended to the analysis of anisotropic solids by using the proper $f(s_{ij})$. Cases involving discontinuous velocity fields can be treated by considering the additional work done on the surface of discontinuity.

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