THE ASSOCIATED ELASTIC PROBLEMS IN DYNAMIC VISCO-ELASTICITY*

BY

L. N. TAO

Illinois Institute of Technology, Chicago, Illinois

Abstract. The paper is concerned with the associated elastic problems of linear visco-elastic bodies under dynamic conditions. By the introduction of an adjunct problem and the use of Laplace transforms, an associated elastic problem is established. This includes the quasi-static study of Lee as a special case. Also, a modification of the adjunct problem leads to a second associated elastic problem, which is in agreement with the correspondence principle.

1. Introduction. In the theory of linear visco-elasticity it is known that for quasi-static cases, i.e. cases with negligible inertia effects, the analysis can be simplified through the use of the solutions of associated elastic problems. Alfrey [1] has demonstrated this for an incompressible visco-elastic body with prescribed surface tractions. Tsien [2] has extended this method to compressible materials with constant Poisson's ratios. However, in general, the Poisson's ratio of a visco-elastic material is time-dependent. This fact restricts the applicability of this kind of approach. Lee [3] has removed this limitation and has shown that for visco-elastic bodies obeying the general linear isotropic stress-strain laws, an associated elastic problem can be established by the use of Laplace transforms. The results available in the extensive literature on the theory of elasticity may therefore be utilized in linear visco-elasticity. This method has been successfully applied to a number of visco-elastic problems.[1] However, the method has not been applied successfully to dynamic problems of visco-elasticity, such as waves and vibrations. An attempt has been made by Read [7], who uses the Fourier transform to discuss the general dynamic problem. This leads to an associated elastic problem with body forces proportional to the displacements. This type of problem is uncommon in the theory of elasticity. Accordingly, the aim of utilizing solutions from the theory of elasticity cannot be achieved satisfactorily. This point has been stressed by Lee [4].

On the other hand, Bland [6] has given correspondence principles for both quasi-static and dynamic problems. The correspondence principle in the quasi-static case was established and based on the associated elastic problem by Lee [3]. A similar situation does not exist for the dynamic case since no dynamic counterpart to the associated elastic problem has yet been found.

The purpose of this paper is first to show that for a visco-elastic body obeying the general linear isotropic stress-strain laws and under dynamic conditions, associated elastic problems of dynamic nature may be found. This is accomplished by introducing an *adjunct* set of equations. This establishment of associated elastic problems contributes to the understanding of the correspondence principle under dynamic conditions. Furthermore, both associated elastic problems and correspondence principles fail to exist when the stresses and/or displacements are initially non-zero. Using the *adjunct* problem, the visco-elastic problem with non-vanishing initial conditions can be analyzed with time-

---

*Received June 26, 1962; revised manuscript received Jan. 31, 1963.

[1]See the articles by Lee [4] and Hunter [5], and the monograph by Bland [6].

independent boundary conditions. This may, to a certain extent, simplify the mathematical analysis.

In the next section, we will first introduce and discuss the *adjunct* problems. Based on the first *adjunct* problem, the first associated elastic problem is then found. As a special case, this associated elastic problem is in complete agreement with the associated elastic problem for the quasi-static case suggested by Lee [3]. In this case, the *adjunct* problem is merely the original visco-elastic problem itself.

In order to present a better understanding of the correspondence principle, a second associated elastic problem, which is a modification of the first associated elastic problem, will be offered. This readily gives the direct relation of the correspondence principle and the associated elastic problem. In general, the choice of the two proposed *adjunct* problems, and hence of the associated elastic problems, depends upon the prescribed boundary conditions and body forces.

2. The adjunct problem. Let us consider a material satisfying the general isotropic linear visco-elastic laws:

$$\begin{aligned} P(D_t)s_{ij} &= Q(D_t)e_{ij} , \\ P'(D_t)\sigma_{ii} &= Q'(D_t)\epsilon_{ii} , \end{aligned} \tag{2.1}$$

where P, Q, P' and Q' are linear operators of $D_t \equiv \partial/\partial t$, σ_{ij} and ϵ_{ij} are the stress and strain tensors, and s_{ij} and e_{ij} their deviators defined by

$$\begin{aligned} s_{ij} &= \sigma_{ij} - \tfrac{1}{3}\sigma_{kk}\delta_{ij} , \\ e_{ij} &= \epsilon_{ij} - \tfrac{1}{3}\epsilon_{kk}\delta_{ij} , \end{aligned} \tag{2.2}$$

where δ_{ij} is the Kronecker delta. These linear visco-elastic laws have been fully discussed in [4, 5, 6]. Let the body be subject to prescribed body forces $f_i(x, t)$ per unit volume,[2] surface tractions $T_i(x, t)$ on the surface S_1, and/or surface displacements $W_i(x, t)$ on the surface S_2 as they may occur in the theory of elasticity. The visco-elastic problem is completely described by the equations of motion

$$\sigma_{ij,j} + f_i(x, t) = \rho D_t^2 u_i \tag{2.3}$$

and the strain-displacement relations

$$\epsilon_{ij} = (u_{i,j} + u_{j,i})/2 \tag{2.4}$$

subject to the boundary conditions

$$\begin{aligned} \sigma_{ij}n_j &= T_i(x, t) \quad \text{on} \quad S_1 , \\ u_i &= W_i(x, t) \quad \text{on} \quad S_2 , \end{aligned} \tag{2.5}$$

and proper initial conditions.

Denote the Laplace transform operator with respect to time t by L_t and the transformed functions by a superscript L on the corresponding functions. Taking the Laplace transforms of the above equations, we then have the visco-elastic problem described by

---

[2]We abbreviate $f_i(x_1, x_2, x_3, t)$ by $f_i(x, t)$.

$$
\begin{aligned}
P(p)s_{ij}^L + P_0(x, p) &= Q(p)e_{ij}^L + Q_0(x, p), \\
P'(p)\sigma_{ii}^L + P_0'(x, p) &= Q'(p)\epsilon_{ii}^L + Q_0'(x, p), \\
\sigma_{ij,j}^L + f_i^L(x, p) &= \rho[p^2 u_i^L + u_i^0(x, p)], \\
\epsilon_{ij}^L &= (u_{i,j}^L + u_{j,i}^L)/2, \\
\sigma_{ij}^L n_j &= T_i^L(x, p) \quad \text{on} \quad S_1 , \\
u_i^L &= W_i^L(x, p) \quad \text{on} \quad S_2 ,
\end{aligned}
\tag{2.6}
$$

where P_0, Q_0, P_0', Q_0' and u_i^0 have obvious meanings and are specified by the initial conditions and the linear operators P, Q, P' and Q'.

Now let us consider an *adjunct* problem: Denote the stress tensors by $\beta_{ij}(x, t, \tau)$, the strain tensors by $\gamma_{ij}(x, t, \tau)$, and their deviators by $\alpha_{ij}(x, t, \tau)$ and $\eta_{ij}(x, t, \tau)$. All of these are functions of the coordinates x_i, the time t, and a new variable τ. Suppose the material obeys a new set of stress-strain laws,

$$
\begin{aligned}
P(D_\tau)\alpha_{ij} &= Q(D_\tau)\eta_{ij} , \\
P'(D_\tau)\beta_{ii} &= Q'(D_\tau)\gamma_{ii} ,
\end{aligned}
\tag{2.7}
$$

where $D_\tau \equiv \partial/\partial\tau$. Let the body be subject to the prescribed body forces $f_i(x, \tau)$, surface tractions $T_i(x, \tau)$ on S_1, and surface displacements $W_i(x, \tau)$ on S_2. All of these prescribed functions are the previous ones with direct replacements of the time t by the new variable τ. The additional equations of the *adjunct* problem are

$$
\begin{aligned}
\beta_{ij,j} + f_i(x, \tau) &= \rho D_t^2 v_i , \\
\gamma_{ij} &= (v_{i,j} + v_{j,i})/2, \\
\beta_{ij}n_j &= T_i(x, \tau) \quad \text{on} \quad S_1 , \\
v_i &= W_i(x, \tau) \quad \text{on} \quad S_2 .
\end{aligned}
\tag{2.8}
$$

Now let us take the double Laplace transforms with respect to τ and t, and denote the transforms by a double star on the functions, We then have

$$
\begin{aligned}
P(p)\alpha_{ij}^{**} + \frac{1}{p} P_0(x, p) &= Q(p)\eta_{ij}^{**} + \frac{1}{p} Q_0(x, p), \\
P'(p)\beta_{ii}^{**} + \frac{1}{p} P_0'(x, p) &= Q'(p)\gamma_{ii}^{**} + \frac{1}{p} Q_0'(x, p), \\
\beta_{ij,j}^{**} + \frac{1}{p} f_i^L(x, p) &= \rho\left[p^2 v_i^{**} + \frac{1}{p} u_i^0(x, p) \right], \\
\gamma_{ij}^{**} &= (v_{i,j}^{**} + v_{j,i}^{**})/2, \\
\beta_{ij}^{**} n_j &= \frac{1}{p} T_i^L(x, p) \quad \text{on} \quad S_1 , \\
v_i^{**} &= \frac{1}{p} W_i^L(x, p) \quad \text{on} \quad S_2 .
\end{aligned}
\tag{2.9}
$$

Clearly, by comparison of this set of equations with the set of equations (2.6), we must have

$$\begin{aligned} u_i^L &= pv_i^{**}, \\ \sigma_{ij}^L &= p\beta_{ij}^{**}, \qquad \epsilon_{ij}^L = p\gamma_{ij}^{**}, \\ s_{ij}^L &= p\alpha_{ij}^{**}, \qquad e_{ij}^L = p\eta_{ij}^{**}. \end{aligned} \tag{2.10}$$

Since v_i^{**} is the double Laplace transform of v_i, the generalized convolution theorem [8] shows that v_i^{**} is also the single Laplace transform of

$$\int_0^t v_i(x, t - \tau, \tau)\, d\tau \tag{2.11}$$

with respect to t. Hence pv_i^{**} must be the Laplace transform L_t of

$$D_t \int_0^t v_i(x, t - \tau, \tau)\, d\tau. \tag{2.12}$$

This shows that

$$u_i = D_t \int_0^t v_i(x, t - \tau, \tau)\, d\tau. \tag{2.13}$$

Using the substitution $\lambda = t - \tau$, we also have

$$u_i = D_t \int_0^t v_i(x, \lambda, t - \lambda)\, d\lambda. \tag{2.14}$$

Similarly, we have

$$\begin{aligned} \sigma_{ij} &= D_t \int_0^t \beta_{ij}(x, t - \tau, \tau)\, d\tau, \\ \epsilon_{ij} &= D_t \int_0^t \gamma_{ij}(x, t - \tau, \tau)\, d\tau, \\ s_{ij} &= D_t \int_0^t \alpha_{ij}(x, t - \tau, \tau)\, d\tau, \\ e_{ij} &= D_t \int_0^t \eta_{ij}(x, t - \tau, \tau)\, d\tau. \end{aligned} \tag{2.15}$$

Therefore, the original visco-elastic problem is directly related to the *adjunct* problem. The latter has time-independent boundary conditions, and hence is easier to handle. This approach may be used when the associated elastic problem discussed in the next two sections does not exist.

3. The first associated elastic problem. We now proceed to establish the first associated elastic problem. This can only be accomplished when the initial stresses and/or displacements are identically zero. This requirement is also imposed by Lee [3] in the quasi-static case and by Bland [6] in his correspondence principle. Since most physical problems originate from a state of rest, the requirement is not a serious one. The vanishing initial conditions readily imply that P_0, Q_0, P_0', Q_0', and u_i^0 of the last section are identically zero.

Let us take the single Laplace transforms with respect to τ, and denote the transforms by a single star on the functions. We then have

$$
\begin{aligned}
P(p)\alpha_{ij}^* &= Q(p)\eta_{ij}^* \ ,\\
P'(p)\beta_{ij}^* &= Q'(p)\gamma_{ij}^* \ ,\\
\beta_{ij,j}^* + f_i^L(x, p) &= \rho D_t^2 v_i^* \ ,\\
\gamma_{ij}^* &= (v_{i,j}^* + v_{j,i}^*)/2,\\
\beta_{ij}^* n_j &= T_i^L(x, p) \quad \text{on} \quad S_1 \ ,\\
v_i^* &= W_i^L(x, p) \quad \text{on} \quad S_2 \ .
\end{aligned}
\tag{3.1}
$$

With proper modifications of the material constants and the boundary conditions as functions of the parameter p, we may readily see that this is an elastic problem of dynamic nature. This will be called the first associated elastic problem. Therefore, the solutions of the *adjunct* problem are the inverse Laplace transforms with respect to τ, denoted as L_τ^{-1}, of the associated elastic problem with proper modifications of material constants and boundary conditions. With known solutions of the *adjunct* problem, those of the original visco-elastic problem may be found through Eqs. (2.13) and (2.15). Hence we have the first theorem which may be symbolically stated as follows:

$$
\begin{bmatrix}\text{visco-elastic}\\ \text{problem}\end{bmatrix} = D_t \int_0^t d\tau \left\{ L_\tau^{-1} \begin{bmatrix}\text{first associated elastic problem}\\ \text{with prescribed } f_i^L,\ T_i^L,\ W_i^L\end{bmatrix}\right\}_{t\to t-\tau} . \tag{3.2}
$$

It is generally understood that the term quasi-static case means that the inertia effects can be neglected in the equations of motion. This is equivalent to the statement that the density ρ is zero. Let us drop the inertia terms of the adjunct problem. We find that the displacement vectors, stress and strain tensors, and their deviators are time-independent, since both prescribed boundary conditions and governing equations are independent of the time t. Then, from Eqs. (2.13) and (2.14), we have

$$
\begin{aligned}
u_i(x, t) &= D_t \int_0^t v_i(x, 0, \tau)\, d\tau\\
&= D_t \int_0^t v_i(x, \lambda, 0)\, d\lambda = v_i(x, t).
\end{aligned}
\tag{3.3}
$$

This shows that $u_i(x, t)$ and $v_i(x, 0, \tau)$ are identical in the quasi-static case. In other words, the adjunct problem is the visco-elastic problem itself, as used by Lee [3].

4. The second associated elastic problem. In the first associated elastic problem discussed in the last section, the boundary conditions and body forces are independent of t. This may limit the availability of information in the theory of elasticity. A modified form is suggested in this section. Consider a second *adjunct* problem described by

$$
\begin{aligned}
P(D_\tau)\alpha_{ij} &= Q(D_\tau)\eta_{ij} \ ,\\
P'(D_\tau)\beta_{ii} &= Q'(D_\tau)\gamma_{ii} \ ,\\
\beta_{ij,j} + f_i(x, t) &= \rho D_t^2 v_i \ ,\\
\gamma_{ij} &= (v_{i,j} + v_{j,i})/2,\\
\beta_{ij} n_j &= T_i(x, t) \quad \text{on} \quad S_1 \ ,\\
v_i &= W_i(x, t) \quad \text{on} \quad S_2 \ .
\end{aligned}
\tag{4.1}
$$

It is noted that in this set of *adjunct* equations the prescribed functions, f_i, T_i and W_i, are functions of the coordinates x_i and the time t, instead of the coordinates x_i and the variable τ as suggested in the first *adjunct* problem.

The double Laplace transforms $L_{t+\tau}$ of this set of equations are the same as those of Eqs. (2.9). Hence, the visco-elastic solutions are still related to those of the second *adjunct* problem by Eqs. (2.13) and (2.15). But the single Laplace transforms L_τ of Eqs. (4.1) lead to

$$\begin{aligned}
P(p)\alpha_{ij}^* &= Q(p)\eta_{ij}^* , \\
P'(p)\beta_{ii}^* &= Q'(p)\gamma_{ii}^* , \\
\beta_{ij,j}^* + \frac{1}{p} f_i(x, t) &= \rho D_t^2 v_i^* , \\
\gamma_{ij}^* &= (v_{i,j}^* + v_{j,i}^*)/2, \\
\beta_{ij}^* n_j &= \frac{1}{p} T_i(x, t) \quad \text{on} \quad S_1 , \\
v_i^* &= \frac{1}{p} W_i(x, t) \quad \text{on} \quad S_2 .
\end{aligned} \tag{4.2}$$

Clearly, this set of equations may be considered as an elastic problem with time-dependent prescribed body forces f_i, surface tractions T_i and surface displacements W_i. This problem will be called the second associated elastic problem. This set of equations, shows that the Laplace transforms of the solutions of the second *adjunct* problem are the corresponding solutions of the second associated elastic problem divided by the parameter p,

$$\begin{aligned}
pv_i^* &= v_i^e , \\
p\beta_{ij}^* &= \beta_{ij}^e , \qquad p\gamma_{ij}^* = \gamma_{ij}^e , \\
p\alpha_{ij}^* &= \alpha_{ij}^e , \qquad p\eta_{ij}^* = \eta_{ij}^e ,
\end{aligned} \tag{4.3}$$

where the superscript e denotes the second associated elastic problem. Since v_i^{**} is the Laplace transform of v_i^* and $u_i^L = pv_i^{**}$, we conclude that

$$u_i = L_t^{-1}\{pL_t[v_i^e/p]\} = L_t^{-1}\{L_t[v_i^e]\}. \tag{4.4}$$

Similarly, we have

$$\begin{aligned}
\sigma_{ij} &= L_t^{-1}\{L_t[\beta_{ij}^e]\}, \qquad \epsilon_{ij} = L_t^{-1}\{L_t[\gamma_{ij}^e]\}, \\
s_{ij} &= L_t^{-1}\{L_t[\alpha_{ij}^e]\}, \qquad e_{ij} = L_t^{-1}\{L_t[\eta_{ij}^e]\}.
\end{aligned} \tag{4.5}$$

Therefore the visco-elastic solutions are the inverse Laplace transforms of the Laplace transforms of the corresponding solutions in the second associated elastic problem. Symbolically, the second theorem is

$$\begin{bmatrix} \text{visco-elastic} \\ \text{problem} \end{bmatrix} = L_t^{-1}\left\{L_t\begin{bmatrix} \text{second associated elastic problem} \\ \text{with prescribed } f_i , T_i , W_i \end{bmatrix}\right\}. \tag{4.6}$$

Using an alternative form,

$$L_t\begin{bmatrix}\text{visco-elastic}\\ \text{problem}\end{bmatrix} = L_t\begin{bmatrix}\text{second associated elastic problem}\\ \text{with prescribed } f_i \text{ , } T_i \text{ , } W_i\end{bmatrix}, \tag{4.7}$$

we have obtained the correspondence principle [6].

In the case of a quasi-static problem, the second associated elastic problem, which has time-dependent body forces and boundary conditions, is unnecessary. We may observe that the Laplace transform L_t of the second associated elastic problem gives the first problem. This readily relates these two associated elastic problems.

5. Hereditary representation of visco-elastic materials. In the previous sections, we have used the stress-strain laws in terms of differential operators. It is known that for visco-elastic materials the hereditary integrals of stress-strain laws are also frequently used. Since these two forms are equivalent, no detail discussions are necessary. Discussion of a simple one-dimensional case is sufficient to make this point clear. The stress-strain relation may be expressed by

$$\int_{-\infty}^{t} J(t - \lambda) \frac{d\sigma(\lambda)}{d\lambda}\, d\lambda = \epsilon(t)$$

or

$$\sigma(t) = \int_{-\infty}^{t} E(t - \lambda) \frac{d\epsilon(\lambda)}{d\lambda}\, d\lambda, \tag{5.1}$$

where $J(t)$ is the creep compliance and $E(t)$ the relaxation modulus. This suggests that in either *adjunct* problem the stress-strain relation should be replaced by

$$\int_{-\infty}^{\tau} J(\tau - \lambda) \frac{\partial\beta(t, \lambda)}{\partial\lambda}\, d\lambda = \gamma(t, \tau)$$

or

$$\beta(t, \tau) = \int_{-\infty}^{\tau} E(\tau - \lambda) \frac{\partial\gamma(t, \lambda)}{\partial\lambda}\, d\lambda. \tag{5.2}$$

The Laplace transforms L_τ of Eqs. (5.2) are

$$\begin{aligned} pJ^*(p)\beta^* &= \gamma^*, \\ \beta^* &= pE^*(p)\gamma^*. \end{aligned} \tag{5.3}$$

These are the proper stress-strain relations for the associated elastic problems.

6. Discussion. In this paper we have established associated elastic problems in dynamic visco-elasticity. These associated elastic problems may be considered as generalizations of those of quasi-static viscoelastic problems [3]. This work also shows that there exists a direct tie between the associated elastic problems and the correspondence principles. Though the present discussion is restricted to the use of Laplace transforms, the same conclusions can be achieved by means of other integral transforms, such as the one-sided Fourier transform used in the correspondence principle of Bland [6].

It has been pointed out by Lee [3] that in quasi-static problems the inverse transform of the associated elastic problem can be easily completed by means of partial fractions. In dynamic problems even with the corresponding associated elastic problem available, the inverse transform may not be as simple as in the quasi-static cases. This complexity arises due to the fact that branch points will generally exist in the transformed plane.

When the stresses and/or displacements are initially non-zero, such as problems of unloading, the associated elastic problem fails to exist. In this case, it seems that the first *adjunct* problem, Equations (2.7) and (2.9), may be used for stress analysis. The main advantages here are that the boundary conditions are time-independent. This is analogous to Duhamel's treatment of the diffusion equation.

References

1. T. Alfrey, *Non-homogeneous stresses in visco-elastic media*, Quart. Appl. Math. **2**, 113-119 (1944)
2. H. S. Tsien, *A generalization of Alfrey's theorem for visco-elastic media*, Quart. Appl. Math. **8**, 104–106 (1950)
3. E. H. Lee, *Stress analysis in visco-elastic bodies*, Quart. Appl. Math. **13**, 183-190 (1955)
4. E. H. Lee, Viscoelastic stress analysis, in *Structural Mechanics*, Pergamon Press, 1960, pp. 456-482
5. S. C. Hunter, Viscoelastic waves, in *Progress in solid mechanics*, vol. I, North-Holland, 1960, pp. 1-57
6. D. R. Bland, *The theory of linear viscoelasticity*, Pergamon Press, 1960
7. W. T. Read, *Stress analysis for compressible visco-elastic materials*, J. Appl. Phys. **21**, 671-674 (1950)
8. R. C. F. Bartels and R. V. Churchill, *Resolution of boundary problems by the use of a generalized convolution*, Bull. Am. Math. Soc. **48**, 276-282 (1942).