PLASTIC PLATE THEORY*

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Abstract. It is shown that the governing equations for perfectly plastic flow of plates are generally elliptic, the only exceptions being certain piecewise linear yield conditions and corners of the yield curve.

Rotationally symmetric plastic plate problems were first investigated by Hopkins and Prager [1] in 1953 and have been extensively treated since then (see, for instance, [2] or [3] for textbook accounts). The general plastic plate problem has been treated theoretically by Hopkins [4] for a particular yield condition, a particular simple problem has been solved by Lerner and Prager [5], and some limit analysis bounds on yield-point loads have been found by Prager and Hodge [6] and improved on by Shull and Hu [7]. However, the complexity of the general problem is such that numerical methods must almost certainly be used if further solutions are to be obtained. An obvious prerequisite of the application of numerical methods is a determination of the type of equations which govern the problem. The present note shows that except for certain special yield conditions, the equations are fully elliptic.

The governing equations for a plastic plate are the equilibrium equations

$$M_{x,x} + M_{xy,y} = V_x , \qquad M_{xy,x} + M_{y,y} = V_y , \tag{1a}$$

$$V_{x,x} + V_{y,y} = -P, \tag{1b}$$

the yield condition†

$$F(M_x , M_{xy} , M_y) = 0, \tag{2}$$

and the flow rule

$$-w_{,xx} = \lambda\, \partial F/\partial M_x , \qquad -w_{,yy} = \lambda\, \partial F/\partial M_y ,$$
$$-w_{,xy} = \tfrac{1}{2}\lambda\, \partial F/\partial M_{xy} . \tag{3}$$

Here M_x, M_{xy}, and M_y are bending moments per unit length, V_x and V_y are shear forces per unit length, w is the velocity, λ is a non-negative scalar unknown, and a comma preceding a subscript indicates partial differentiation with respect to the following subscripts.

The shear force may be eliminated by the introduction of a "shear potential" defined by

$$V_x = \Phi_{,y} - \tfrac{1}{2} \int P\, dx, \qquad V_y = -\Phi_{,x} - \tfrac{1}{2} \int P\, dy, \tag{4}$$

whence Eq. (1b) is satisfied identically and (1a) becomes

$$M_{x,x} + M_{xy,y} - \Phi_{,y} = -\tfrac{1}{2} \int P\, dx,$$
$$M_{xy,x} + M_{yy,y} + \Phi_{,x} = -\tfrac{1}{2} \int P\, dy. \tag{5}$$

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†Singular yield conditions and associated flow rules will be considered later.

Next, we introduce new variables defined by

$$M_x = M_0(\omega + \chi \cos 2\theta), \qquad M_y = M_0(\omega - \chi \cos 2\theta),$$
$$M_{xy} = M_0\chi \sin 2\theta,$$

where M_0 is the uniaxial yield moment, ω and χ the dimensionless sum and difference of the principal moments, and θ the inclination of the principal moments to the coordinate axes. For an isotropic material, the yield condition (2) can then be written

$$F(M_x , M_{xy} , M_y) = \chi - f(\omega) = 0. \tag{7}$$

Substitution of (6) and (7) into the equilibrium equations (5) and flow law (3) then leads to

$$\begin{aligned}(1 + f' \cos 2\theta)\omega_{,x} + (f' \sin 2\theta)\omega_{,y} - (2f \sin 2\theta)\theta_{,x} + (2f \cos 2\theta)\theta_{,y} - \phi_{,y} &= p_x , \\ (f' \sin 2\theta)\omega_{,x} + (1 - f' \cos 2\theta)\omega_{,y} + (2f \cos 2\theta)\theta_{,x} + (2f \sin 2\theta)\theta_{,y} + \phi_{,x} &= p_y ,\end{aligned} \tag{8}$$

$$\begin{aligned}w_{,xx} = \mu(f' - \cos 2\theta), \qquad w_{,yy} = \mu(f' + \cos 2\theta), \\ w_{,xy} = -\mu \sin 2\theta,\end{aligned} \tag{9}$$

where

$$\begin{aligned}f' = df/d\omega, \qquad \mu = \lambda/2M_0 , \qquad \phi = \Phi/M_0 , \\ p_x = -\frac{1}{2M_0}\int P\, dx, \qquad p_y = -\frac{1}{2M_0}\int P\, dy.\end{aligned} \tag{10}$$

Finally, we eliminate w from (9) to obtain the two equations

$$\begin{aligned}f''\omega_{,x} - (2 \sin 2\theta)\theta_{,x} + (2 \cos 2\theta)\theta_{,y} + (f' + \cos 2\theta)\Lambda_{,x} + (\sin 2\theta)\Lambda_{,y} &= 0, \\ f''\omega_{,y} + (2 \cos 2\theta)\theta_{,x} + (2 \sin 2\theta)\theta_{,y} + (\sin 2\theta)\Lambda_{,x} + (f' - \cos 2\theta)\Lambda_{,y} &= 0,\end{aligned} \tag{11}$$

where $\Lambda = \log \mu$.

Equations (8) and (11) are a set of four quasilinear first order equations for ω, θ, ϕ, and Λ. We shall show that they are of fully elliptic type except for certain special yield conditions by showing that their characteristics are all imaginary. Since the type of a set of equations is independent of the choice of coordinate axes, we may first simplify the equations by taking $\theta = 0$, corresponding to a choice of principal directions as coordinate axes. The characteristic curves are then defined by the vanishing of the eight by eight determinant

$$\begin{vmatrix} 1 + f' & 0 & 0 & 2f & 0 & -1 & 0 & 0 \\ 0 & 1 - f' & 2f & 0 & 1 & 0 & 0 & 0 \\ f'' & 0 & 0 & 2 & 0 & 0 & 1 + f' & 0 \\ 0 & f'' & 2 & 0 & 0 & 0 & 0 & -(1 - f') \\ dx & dy & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & dx & dy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & dx & dy & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & dx & dy \end{vmatrix} = 0 \tag{12}$$

Elementary manipulation shows that (12) is equivalent to

$$(1 + f')^2(dy/dx)^4 - 2(2ff'' + f'^2 - 1)(dy/dx)^2 + (1 - f')^2 = 0 \tag{13}$$

If the characteristics are to be real, then the roots of (13), considered as a quadratic equation in $(dy/dx)^2$ must be real and positive. They will be real if

$$4ff''(ff'' + f'^2 - 1) \geq 0. \tag{14}$$

The yield curve must be convex, hence either it is piecewise linear or

$$ff'' < 0. \tag{15}$$

Leaving the linear yield condition for later consideration, we see that (14) becomes

$$ff'' + f'^2 - 1 \leq 0 \tag{16}$$

as a necessary condition for real characteristics.

It is evident from (13) that both roots are of the same sign and that a necessary condition for them to be positive is that

$$2ff'' + f'^2 - 1 \geq 0. \tag{17}$$

However, in view of (15), (17) is incompatible with (16) so that dy/dx is never real and the equations are elliptic.

For a piecewise linear yield condition

$$f = a\omega + b \tag{18}$$

and the two roots of (13) coincide at

$$\left(\frac{dy}{dx}\right)^2 = \frac{a - 1}{a + 1}. \tag{19}$$

Thus, if $|a| < 1$, the equations are elliptic; if $|a| > 1$, they have two real and distinct double characteristics; and if $|a| = 1$, they have a single quadruple characteristic.

Finally, we consider a corner of the yield curve where f' is not uniquely defined. Let

$$\chi = f_1(\omega), \qquad \chi = f_2(\omega) \tag{20}$$

be the equations of the two sides which form the corner. Equations (20) uniquely determine $\omega = \omega_0$, $\chi = \chi_0$ so that (8) becomes

$$\begin{aligned} -(2\chi_0 \sin 2\theta)\theta_{,x} + (2\chi_0 \cos 2\theta)\theta_{,y} - \phi_{,y} &= p_x, \\ (2\chi_0 \cos 2\theta)\theta_{,x} + (2\chi_0 \sin 2\theta)\theta_{,y} + \phi_{,x} &= p_y. \end{aligned} \tag{21}$$

Equations (21) are a pair of equations for θ and ϕ whose characteristics are the real curves

$$dy/dx = \tan\theta, \qquad dy/dx = -\cot\theta, \tag{22}$$

i.e., the lines of principal direction. The flow law (9) for a corner must be replaced by

$$\begin{aligned} w_{,xx} &= \mu_1(f_1' - \cos 2\theta) + \mu_2(f_2' - \cos 2\theta), \\ w_{,yy} &= \mu_1(f_1' + \cos 2\theta) + \mu_2(f_2' + \cos 2\theta), \\ w_{,xy} &= -(\mu_1 + \mu_2)\sin 2\theta. \end{aligned} \tag{23}$$

Elimination of μ_1 and μ_2 from (23) yields the equation

$$w_{,xx} - 2 \cot 2\theta w_{,xy} - w_{,yy} = 0, \tag{24}$$

whose characteristics are again (22). Thus, in this case also there are two real distinct double characteristics for the problem.

An exception to the preceding paragraph occurs when $\chi_0 = 0$. In this case the two principal moments are equal so that θ is not defined and, in fact, any curve may be considered a characteristic.

For the particular case of the Tresca yield condition of maximum shearing stress, it may readily be verified that the results obtained above for piecewise linear yield curves and for corners on the yield curve agree with the conclusions previously reached by Hopkins [4] by a somewhat different line of reasoning.

References

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