THE MAXIMUM-MINIMUM PRINCIPLES FOR A QUASI-LINEAR PARABOLIC FINITE DIFFERENCE EQUATION*

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Abstract. A strong maximum principle for second order parabolic equations has been introduced by L. Nirenberg. The present paper contains both strong and weak maximum-minimum principles for various finite difference equations which approximate a quasi-linear parabolic differential equation. A proof of the existence and uniqueness of the solution of the finite difference equations is also presented.

1. Introduction. In the study of one-dimensional longitudinal impact on visco-plastic rods [1] one encounters the following second order quasi-linear parabolic equation

$$\frac{\partial^2 \sigma}{\partial x^2} - \frac{\partial}{\partial t} F(\sigma) = 0 \tag{1}$$

where $F(\sigma)$ is a continuous increasing function of σ with piece-wise continuous first derivative. Physically, $\sigma(x, t)$ represents the nominal stress, and $F(\sigma)$ the corresponding strain rate. The following stress-strain rate relation, for instance, is the simplest one for visco-plastic materials:

$$\begin{aligned} F(\sigma) &= \sigma - k && \text{if} \quad \sigma > k, \\ F(\sigma) &= 0 && \text{if} \quad |\sigma| \leq k, \\ F(\sigma) &= -\sigma + k && \text{if} \quad \sigma < -k; \end{aligned}$$

here k is a constant, the yield stress. Equation (1) then is equivalent to the following set of equations:

$$\begin{aligned} \frac{\partial^2 \sigma}{\partial x^2} - \frac{\partial \sigma}{\partial t} &= 0 && \text{if} \quad |\sigma| > k, \\ \frac{\partial^2 \sigma}{\partial x^2} &= 0 && \text{if} \quad |\sigma| \leq k. \end{aligned}$$

In this paper, the term *function* will mean a single-valued function. By a *monotonically increasing function* (or simply *increasing function*) we indicate that for given $\sigma^{(1)}$ and $\sigma^{(2)}$, the inequality $\sigma^{(1)} > \sigma^{(2)}$ implies $F(\sigma^{(1)}) \geq F(\sigma^{(2)})$. Similarly, the term *strictly increasing function* means that $\sigma^{(1)} > \sigma^{(2)}$ implies $F(\sigma^{(1)}) > F(\sigma^{(2)})$. Hence, in the region where $F(\sigma)$ is strictly increasing with continuous first derivative, we can write Eq. (1) as

$$\frac{\partial^2 \sigma}{\partial x^2} - f(\sigma) \frac{\partial \sigma}{\partial t} = 0, \qquad f(\sigma) > 0. \tag{2}$$

Consider a simple domain D in the x, t plane bounded by lines $x = 0$, $x = l$, $t = 0$ and $t = t_0$ (Fig. 1). In addition, denote the top boundary ab (excluding the end points)

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by B_0 and the rest of the boundary $ao + oc + cb$ by B_1. Therefore, $B = B_0 + B_1$ is the entire boundary of D.

If a function $\sigma(x, t)$ which is defined and continuous in $D + B$ satisfies Eq. (2), we can write the maximum-minimum principles as follows.

Theorem I (*weak* principle). Every solution $\sigma(x, t)$ of (2) defined and continuous in $D + B$ attains its maximum and minimum values on B_1.

Theorem II (*strong* principle). Let $\sigma(x, t)$ be a solution of (2) defined and continuous in $D + B$. If $\sigma(x, t)$ attains its maximum (or minimum) value on B_0, then $\sigma(x, t) \equiv$ constant in $D + B$.

It is evident that the weak principle is a consequence of the strong principle. The strong principle has been proved by Nirenberg [2] for linear second order parabolic equations. The proof given by Nirenberg is applicable to certain non-linear parabolic equations such as (2). For the weak principle, a simpler proof may be obtained by slightly modifying the proof for the case of linear parabolic equations (see, for instance, [3]).

In practical problems, $f(\sigma)$ is not in general a constant, and one has to solve the non-linear equation (2). Except in a special case [4], analytic solution of (2) is very difficult, if not impossible. Numerical solution by finite difference methods is one way to solve Eq. (2) approximately. Suppose one has replaced Eq. (2) by a finite difference equation and solved the latter in the region $D + B$ for the values of $\sigma(x, t)$ at discrete grid points. We may then ask whether the maximum-minimum principles (Theorems I and II) still hold in $D + B$. The weak principle for linear finite difference equations has been introduced by Forsythe and Wasow [5]. In Sect. 2, we shall investigate both strong and weak principles for the non-linear finite difference equations which approximate Eq. (1). Uniqueness and existence of the finite difference equations are presented in Sect. 3.

2. Finite difference equations. We will investigate the finite difference approximation to equation (1). Here $F(\sigma)$ is a continuous increasing function of σ (not necessarily strictly increasing) with piece-wise continuous first derivative.

Since the solution of a finite difference equation is evaluated at discrete points, we will redefine the domain D and the boundaries B_0 and B_1. Suppose we take n equal intervals between $x = 0$ and $x = l$, and m equal intervals between $t = 0$ and $t = t_0$ so that the coordinates (x, t) of a generic grid point will be $(i\,\Delta x, j\,\Delta t)$, where $0 \leq i \leq n$, $0 \leq j \leq m$. For simplicity, a point $(i\,\Delta x, j\,\Delta t)$ will be denoted by (i, j) and the stress $\sigma(i\,\Delta x, j\,\Delta t)$ at this point by $\sigma_{i,j}$. The domain D will include all points (i, j) for which $0 < i < n$, $0 < j < m$. The boundary B_0 will have points (i, m) where $0 < i < n$ and the boundary B_1 will have points $(0, j)$, (n, j) and $(i, 0)$ where $0 < j \leq m$, $0 \leq i \leq n$. As before, $B = B_0 + B_1$.

The finite difference equations which approximate Eq. (1) have the following forms.

Forward method:

$$r[\sigma_{i-1,j-1} - 2\sigma_{i,j-1} + \sigma_{i+1,j-1}] = F(\sigma_{i,j}) - F(\sigma_{i,j-1}); \tag{3}$$

backward method:

$$r[\sigma_{i-1,j} - 2\sigma_{i,j} + \sigma_{i+1,j}] = F(\sigma_{i,j}) - F(\sigma_{i,j-1}); \tag{4}$$

Crank-Nicolson method:

$$\frac{r}{2}\,[(\sigma_{i-1,j-1} - 2\sigma_{i,j-1} + \sigma_{i+1,j-1}) + (\sigma_{i-1,j} - 2\sigma_{i,j} + \sigma_{i+1,j})] = F(\sigma_{i,j}) - F(\sigma_{i,j-1}), \tag{5}$$

where $r = \Delta t/(\Delta x)^2$ is the mesh ratio. More generally, if $0 \leq \lambda \leq 1$, one can combine Eqs. (3), (4) and (5) into one equation:

$$r\lambda[\sigma_{i-1,j-1} - 2\sigma_{i,j-1} + \sigma_{i+1,j-1}] + r(1-\lambda)[\sigma_{i-1,j} - 2\sigma_{i,j} + \sigma_{i+1,j}] = F(\sigma_{i,j}) - F(\sigma_{i,j-1}). \tag{6}$$

The common right hand side of Eqs. (3) to (6) can be written as

$$F(\sigma_{i,j}) - F(\sigma_{i,j-1}) = f_{i,j}[\sigma_{i,j} - \sigma_{i,j-1}],$$

where

$$f_{i,j} = \frac{F(\sigma_{i,j}) - F(\sigma_{i,j-1})}{\sigma_{i,j} - \sigma_{i,j-1}} \geq 0 \tag{7}$$

by virtue of the monotonicity of $F(\sigma)$. In the case when $dF/d\sigma$ is continuous for the value of σ between $\sigma_{i,j}$ and $\sigma_{i,j-1}$, $f_{i,j} = f(\sigma^*)$ where σ^* has a value between $\sigma_{i,j}$ and $\sigma_{i,j-1}$. Thus, after rearranging, Eq. (6) can be written as

$$\sigma_{i,j} = \frac{1}{f_{i,j} + 2r(1-\lambda)} [r\lambda(\sigma_{i-1,j-1} + \sigma_{i+1,j+1}) + r(1-\lambda)(\sigma_{i-1,j} + \sigma_{i+1,j}) + (f_{i,j} - 2r\lambda)\sigma_{i,j-1}],$$

$$f_{i,j} \geq 0, \qquad 0 < i < n, \qquad 0 < j \leq m. \tag{8}$$

The proof of the maximum-minimum principles depends on the following lemma.

Lemma I. Let $u_0, u_1, u_2, \cdots, u_n$ be $n + 1$ real-valued quantities related by

$$u_0 = c_1u_1 + c_2u_2 + \cdots + c_nu_n, \tag{9}$$

where c_i are non-zero positive coefficients such that

$$c_1 + c_2 + \cdots + c_n = 1. \tag{10}$$

Then, if u_0 is the maximum of u_i $(0 \leq i \leq n)$, $u_0 = u_1 = u_2 = \cdots = u_n$. The same result applies if u_0 is the minimum of u_i. In other words, u_0 cannot be the maximum or minimum value of u_i unless they are all identical.

The proof is simple. Assume that the lemma is not true and u_0 is the maximum (or minimum) of u_i. Making use of Eq. (10), Eq. (9) can be written as

$$(c_1 + c_2 + \cdots + c_n)u_0 = c_1u_1 + c_2u_2 + \cdots + c_nu_n$$

or

$$c_1(u_0 - u_1) + c_2(u_0 - u_2) + \cdots + c_n(u_0 - u_n) = 0.$$

Since c_i are non-zero positives and u_0 is the maximum (or minimum) of u_i, the left hand side of the equality cannot vanish unless $u_0 = u_1 = \cdots = u_n$.

Theorem III. Every solution of Eq. (8) in $D + B$ satisfies the strong principle if:

(a) $$0 < \lambda < 1$$

and

(b) $$r < \frac{1}{2\lambda} f_{i,j} \quad \text{for all} \quad 0 < i < n, \qquad 0 < j \leq m.$$

In other words, if conditions (a) and (b) are satisfied, the assumption that $\sigma_{i,j}$ attains its maximum (or minimum) on B_0 implies $\sigma_{i,j} \equiv$ constant in $D + B$.

Proof. With conditions (a) and (b), the coefficients of the terms on the right hand side of Eq. (8) are non-zero positives and the sum of the coefficients is equal to one. Therefore Lemma I is applicable to Eq. (8).

Suppose that the maximum (or minimum) of $\sigma_{i,j}$ is $\sigma_{q,m}$ $(0 < q < n)$ at the point (q, m) on B_0. Then, by Lemma I, $\sigma_{q,m} = \sigma_{q-1,m} = \sigma_{q+1,m} = \sigma_{q,m-1} = \sigma_{q-1,m-1} = \sigma_{q+1,m-1}$. Since $\sigma_{q-1,m}$ is also the maximum (or minimum) of $\sigma_{i,j}$, by Lemma I again $\sigma_{q-1,m} = \sigma_{q-2,m} = \sigma_{q-2,m-1}$. Repeating this process on both sides of (q, m) and points below (q, m), it is easily seen that all $\sigma_{i,j}$ are identical in $D + B$.

It should be noticed that condition (b) is sufficient but not necessary. The theorem is still true if $r = f_{i,j}/2\lambda$. For this case, the coefficient of the last term of Eq. (8) vanishes and the assumption that $\sigma_{q,m}$ is the maximum leads to the result $\sigma_{q,m} = \sigma_{q-1,m} = \sigma_{q+1,m} = \sigma_{q-1,m-1} = \sigma_{q+1,m-1}$ and nothing can be said about $\sigma_{q,m-1}$. However, since $\sigma_{q-1,m}$ is also the maximum, by Lemma I we find that $\sigma_{q,m-1} = \sigma_{q,m}$. The proof fails if (q, m) is the only point of B_0. Thus, we have the following theorem.

Theorem III′. If $0 < \lambda < 1$, $n > 2$ and $r \leq f_{i,j}/2\lambda$ for all $0 < i < n$, $0 < j \leq m$, the assumption that $\sigma_{i,j}$ attains its maximum (or minimum) on B_0 implies $\sigma_{i,j} \equiv$ constant in $D + B$.

The next two theorems state the maximum-minimum principles for $\lambda = 0$ (the backward difference method) and $\lambda = 1$ (the forward difference method). The proofs are similar to that of Theorem III and are omitted here.

Theorem IV. If $\lambda = 0$, and for each j $(j = 1, 2, \cdots m)$ at least one of $f_{i,j}$ $(0 < i < n)$ is non-zero, the assumption that $\sigma_{i,j}$ attains its maximum (or minimum) on B_0 implies $\sigma_{i,j}$ are identical in $D + B$ except possibly $\sigma_{0,0}$ and $\sigma_{n,0}$.

Theorem V. If $\lambda = 1$ and $r \leq f_{i,j}/2$ for all $0 < i < n$, $0 \leq j < m$, $\sigma_{i,j}$ attains its maximum and minimum values on B_1 (weak principle).

The results of Theorems III to V are summarized in Table I. For all $0 < \lambda \leq 1$

TABLE I

	weak principle	strong principle	stability for linear parabolic equation ($f_{i,j} \equiv 1$)
$\lambda = 1$, forward method	$r \leq f_{i,j}/2$	no r	$r \leq \frac{1}{2}$
$\lambda = 0$, backward method	all r	all r (except $\sigma_{0,0}$ and $\sigma_{n,0}$) see Theorem IV for restriction on $f_{i,j}$	all r
$\lambda = \frac{1}{2}$ Crank-Nicolson method	$r \leq f_{i,j}$	$r < f_{i,j}$, if $n = 2$ $r \leq f_{i,j}$, if $n > 2$	all r

there is a restriction on the mesh ratio r so that the maximum-minimum principles are satisfied. As a comparison, the stability criteria on the mesh ratio for the linear parabolic equation are listed in the last column. It is interesting to see that while the Crank-Nicolson method is stable for all r, the maximum-minimum principles hold only for $r \leq 1$ (in linear case $f_{i,j} \equiv 1$). In other words, the maximum-minimum principles imply the stability of the solution. The converse, however, is not necessarily true.

Finally, we state without proof a theorem which is a counterpart of a result derived Pólya and Szegö [6] for parabolic differential equations.

Theorem VI. Let $\sigma_{i,j}$ be a solution of Eq. (8) with $\lambda = 0$ (backward difference method). For a fixed j, let the maximum value of $\sigma_{i,j}$ $(0 \leq i \leq n)$ be $\sigma_{\alpha,j}$ say. Similarly, let the maximum value of $\sigma_{i,j-1}$ $(0 \leq i \leq n)$ be $\sigma_{\beta,j-1}$. If $0 < \alpha < 1$, $0 < \beta < 1$, then

$$\sigma_{\alpha,j} \leq \sigma_{\beta,j-1} .$$

Furthermore, if at least one of $f_{i,j}$ $(i = 1, 2, \cdots, n - 1)$ is non-zero, then the equality holds only if $\sigma_{i,j}$ and $\sigma_{i,j-1}$ are identical for all $0 \leq i \leq n$.

This theorem states that for each time t, if the maximum value occurs in D, it will decrease its magnitude as time t increases. Similarly, one can easily modify Theorem VI such that for each t if the minimum value occurs in D, its magnitude will increase as time t increases.

Theorem VI states the result of Pólya and Szegö for the backward difference equation. Clearly, it is easily modified to hold for other difference equations.

In closing this section, we remark that all the theorems stated hold equally well for a region with moving boundaries. The boundaries oa and cb (Fig. 1) can be any curves

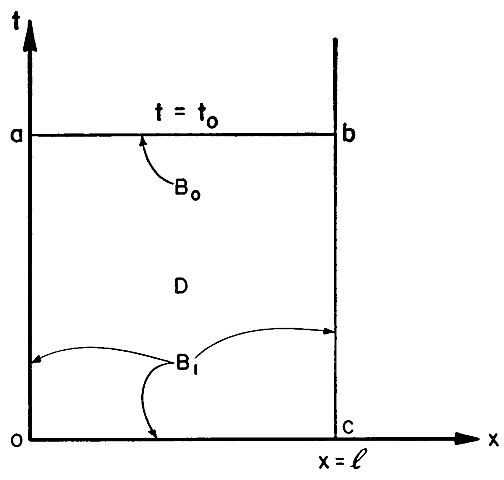

FIG. 1

in x-t plane provided that for each line $t = t^*$ say, the moving boundaries cross the line $t = t^*$ only once. We further remark that the maximum-minimum principles are also applicable to more general parabolic finite difference equations if their finite difference equations satisfy the conditions of Lemma I. Not all of the parabolic finite difference equations satisfy the conditions of Lemma I. The following lemma will be found useful when Lemma I is not applicable.

Lemma II. Let $u_0, u_1, u_2, \cdots, u_n$ be $n + 1$ real-valued quantities related by

$$eu_0 = c_1u_1 + c_2u_2 + \cdots + c_nu_n + d$$

where c_i are non-zero positive coefficients such that

$$c_1 + c_2 + \cdots + c_n = 1.$$

If $e \geq 1$ and $d \leq 0$, $u_0 \leq \max [u_i]$, $(0 \leq i \leq n)$. Also, if $0 < e \leq 1$ and $d \geq 0$, $u_0 \geq \min [u_i]$, $(0 \leq i \leq n)$.

Proof. By Lemma I, one obtains

$$\min [u_i] \leq eu_0 - d \leq \max [u_i].$$

If $e \geq 1$ and $d \leq 0$,

$$u_0 \leq \frac{1}{e} \max [u_i] + \frac{d}{e} \leq \max [u_i].$$

Similarly, if $0 < e \leq 1$ and $d \geq 0$,

$$u_0 \geq \frac{1}{e} \min [u_i] + \frac{d}{e} \geq \min [u_i].$$

3. Solution of the finite difference equations. The simplest boundary value problem for Eq. (1) is to specify the values σ at $x = 0$ and $x = 1$. In the finite difference equations, this is expressed by

$$\begin{aligned} \sigma_{0,j} &= a_j, \\ \sigma_{n,j} &= b_j, \qquad (j = 1, 2, \cdots m), \end{aligned} \tag{11a}$$

where a_j and b_j are known. More generally, a boundary condition for Eq. (1) may involve a relation between σ, $\partial\sigma/\partial x$ and $\partial\sigma/\partial t$. This relation may be linear or non-linear. After replacing the partial derivatives by some finite difference forms, we assume that the following boundary conditions are obtained:

$$\begin{aligned} \sigma_{1,j} - \sigma_{0,j} &= P_j(\sigma_{0,j}, \sigma_{q,j-1}), \\ \sigma_{n-1,j} - \sigma_{n,j} &= Q_j(\sigma_{n,j}, \sigma_{q,j-1}), \qquad (j = 1, 2, \cdots m), \end{aligned} \tag{11b}$$

where P_j is a function of $\sigma_{0,j}$ and $\sigma_{q,j-1}$ $(q = 0, 1, 2 \cdots n)$. Similarly, Q_j is a function of $\sigma_{n,j}$ and $\sigma_{q,j-1}$.

The initial condition in the finite difference form is

$$\sigma_{i,0} = c_i, \qquad (i = 0, 1 \cdots n). \tag{12}$$

If $\lambda \neq 1$, Eq. (8) or Eq. (6) which approximates Eq. (1) can be written as

$$\sigma_{i+1,j} - 2\sigma_{i,j} + \sigma_{i-1,j} = \Phi_{i,j}(\sigma_{i,j}, \sigma_{q,j-1}), \qquad (0 < i < n, 0 < j \leq m), \tag{13a}$$

where

$$\Phi_{i,j}(\sigma_{i,j}, \sigma_{q,j-1}) = \frac{1}{r(1-\lambda)} [F(\sigma_{i,j}) - F(\sigma_{i,j-1})]$$

$$- \frac{\lambda}{1-\lambda} [\sigma_{i-1,j-1} - 2\sigma_{i,j-1} + \sigma_{i,j-1}], \qquad 0 \leq \lambda < 1. \tag{13b}$$

Thus, the system of equations (13) with the initial condition (12) and the boundary conditions (11a) or (11b) give a complete description of a parabolic finite difference equation.

The existence and uniqueness of the solution of this parabolic equation is equivalent to the existence and uniqueness of the solution $\sigma_{i,j}$ ($i = 0, 1, \cdots n$) of Eqs. (13) with (11a) or (11b) when $\sigma_{i,j-1}$ ($i = 0, 1, \cdots n$) are known. For simplicity, the existence and uniqueness of Eqs. (13) and (11b) will be proved in the following form.

Theorem VII (uniqueness and existence). The system of equations

$$\begin{aligned} u_1 - u_0 \qquad\quad &= \varphi_0(u_0), \\ u_2 - 2u_1 + u_0 &= \varphi_1(u_1), \\ u_3 - 2u_2 + u_1 &= \varphi_2(u_2), \\ \cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots \\ u_n - 2u_{n-1} + u_{n-2} &= \varphi_{n-1}(u_{n-1}), \\ -u_n + u_{n-1} &= \varphi_n(u_n), \end{aligned} \tag{14}$$

has a unique solution if

(a) φ_i $(0 \leq i \leq n)$ are continuous increasing functions of u_i, i.e. if $\Delta u_i = u_i^{(1)} - u_i^{(2)} > 0$, $\Delta\varphi_i = \varphi_i(u_i^{(1)}) - \varphi_i(u_i^{(2)}) \geq 0$, and

(b) there exists a positive number $M > 0$ such that at least one of φ_i, say φ_k, satisfies the condition

$$\Delta\varphi_k \geq M\Delta u_k \quad \text{for all} \quad \Delta u_k > 0.$$

The following proof of the theorem constitutes a method of interation for the numerical solution of Eqs. (14), though it may not be a practical one. Suppose that one assumes an approximate value $u_0^{(1)}$ for u_0 and determines $u_1^{(1)}$ from the first equation of (14). The second equation of (14) determines $u_2^{(1)}$ and the third equation $u_3^{(1)}$ and so on. Finally, the penultimate equation of (14) furnishes $u_n^{(1)}$ and we determine $\psi^{(1)}$ by

$$\psi = u_n - u_{n-1} + \varphi_n(u_n). \tag{15}$$

If $\psi^{(1)}$ so obtained happens to be zero, $u_0^{(1)}, u_1^{(1)}, \cdots u_n^{(1)}$ are a solution of (14). Otherwise we assume another approximate value $u_0^{(2)}$ for u_0 and repeat the preceding procedure to find $\psi^{(2)}$. We assert that: (i) ψ is a continuous and strictly increasing function of u_0, i.e., if $\Delta u_0 = u_0^{(1)} - u_0^{(2)} > 0$, $\Delta\psi = \psi^{(1)} - \psi^{(2)} > 0$, and (ii) there exists a positive number $N \geq M$ such that $\Delta\psi \geq N\,\Delta u_0$ for all $\Delta u_0 > 0$.

It is clear that ψ is a continuous function of u_0. Now, if $\Delta u_0 = u_0^{(1)} - u_0^{(2)} > 0$, the first n equations of (14) and Eq. (15) give:

$$\begin{aligned} (\Delta u_1 - \Delta u_0) \qquad\qquad\qquad\quad &= \Delta\varphi_0 \geq 0, \\ (\Delta u_2 - \Delta u_1) - (\Delta u_1 - \Delta u_0) &= \Delta\varphi_1 \geq 0, \\ (\Delta u_3 - \Delta u_2) - (\Delta u_2 - \Delta u_1) &= \Delta\varphi_2 \geq 0, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots&\cdots\cdots\cdots \\ (\Delta u_n - \Delta u_{n-1}) - (\Delta u_{n-1} - \Delta u_{n-2}) &= \Delta\varphi_{n-1} \geq 0, \\ \Delta\psi - (\Delta u_n - \Delta u_{n-1}) &= \Delta\varphi_n \geq 0, \end{aligned} \tag{16}$$

where, by hypothesis (a), $\Delta\varphi_i \geq 0$. An immediate result from (16) is

$$\Delta u_n \geq \Delta u_{n-1} \geq \cdots \geq \Delta u_1 \geq \Delta u_0 > 0. \tag{17}$$

Moreover, by adding up the equations in (16) and making use of hypotheses (a), (b) and the inequality (17), one obtains

$$\Delta\psi = \sum_{i=0}^{n} \Delta\varphi_i \geq \Delta\varphi_k \geq M\Delta u_k \geq M\Delta u_0 > 0.$$

This verifies our assertions (i) and (ii).

Now, if $u_i^{(1)}$ and $u_i^{(2)}$ $(i = 0, 1, \cdots, n)$ are both solutions of Eqs. (14), one would have $\psi(u_0^{(1)}) = \psi(u_0^{(2)}) = 0$. Since ψ is a strictly increasing function of u_0, by assertion (i), we have $u_0^{(1)} = u_0^{(2)}$. The solution is unique.

To prove existence, suppose a trial value $u_0^{(1)}$ gives $\psi(u_0^{(1)}) > 0$. By assertion (ii) there exists a positive number $N \geq M > 0$ such that for all $u_0^{(2)}$,

$$\psi(u_0^{(1)}) - \psi(u_0^{(2)}) > N(u_0^{(1)} - u_0^{(2)}),$$

or

$$\psi(u_0^{(2)}) < \psi(u_0^{(1)}) - N(u_0^{(1)} - u_0^{(2)}).$$

In particular, one can choose $u_0^{(2)} < u_0^{(1)}$ such that $\psi(u_0^{(2)}) < 0$. Since ψ is a continuous strictly increasing function of u_0, there exists $u_0^{(2)} < u_0^* < u_0^{(1)}$ such that $\psi(u_0^*) = 0$. Thus the proof is completed.

It should be noticed that Eqs. (13) with the boundary condition (11a) can be written in the form of (14). Since $\sigma_{0,j}$ and $\sigma_{n,j}$ are known, Eqs. (13) can be written as

$$\begin{aligned}
\sigma_{2,j} - \sigma_{1,j} &= \Phi_{1,j} + \sigma_{1,j} + a_j\,,\\
\sigma_{3,j} - 2\sigma_{2,j} + \sigma_{1,j} &= \Phi_{2,j}\,,\\
\sigma_{4,j} - 2\sigma_{3,j} + \sigma_{2,j} &= \Phi_{3,j}\,,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\\
\sigma_{n-1,j} - 2\sigma_{n-2,j} + \sigma_{n-3,j} &= \Phi_{n-2,j}\\
-\sigma_{n-1,j} + \sigma_{n-2,j} &= \Phi_{n-1,j} + \sigma_{n-1,j} + b_j\,.
\end{aligned}$$

If $\Phi_{i,j}$ $(i = 1, 2, \cdots, n-1)$ are continuous increasing functions of $\sigma_{i,j}$, condition (a) of Theorem VII is satisfied. Moreover, condition (b) is also satisfied since the right hand side of the first equation has the relation,

$$\Delta(\Phi_{1,j} + \sigma_{1,j} + a_j) \geq \Delta\sigma_{1,j}\,.$$

Hence, for the boundary condition (11a) we have the following less restrictive theorem:

Theorem VIII. The parabolic finite difference equation (13) with the boundary condition (11a) and the initial condition (12) has a unique solution if F is a continuous increasing function of its argument.

Finally, it is obvious that Theorem VII can be applied to mixed boundary value problems. For instance, if the condition at $x = 0$ is specified by the first equation of (11a) and the condition at $x = l$ is specified by the second equation of (11b), then Eq. (13) has a unique solution when F is a continuous increasing function. For more complicated mixed boundary value problems, a more strict condition (condition (b) of Theorem VII) may be required on function F.

4. Conclusions. One may prove the uniqueness part of Theorem VIII by employing the maximum-minimum principles. It is well known that the maximum-minimum principles are used frequently in the proof of uniqueness. The proof of uniqueness with the aid of the maximum-minimum principles is particularly simple if the equations are linear and the boundary values are specified by (11a). For non-linear difference equations with mixed boundary values as represented by Eq. (11b), the usefulness of the maximum-minimum principles in the uniqueness proof is limited. However, the importance of the maximum-minimum principles is not restricted to the proof of uniqueness. The principles can be used to explain and prove many physically significant properties. In the problems of visco-plastic impact on thin rods, for example, by using the maximum-minimum principles one can predict the shape of the unloading boundary without explicity solving the moving boundary value problem. Therefore, for numerical solutions of physical problems, one must choose the mesh ratio such that (in addition to ensuring convergence and stability) the maximum-minimum principles are not violated; otherwise the requirements of the original differential equations would not be satisfied.

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