THE MATRIX OF A TRANSFORMERLESS NETWORK*

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1. Introduction. The electric networks under consideration in this paper are bilateral, lumped, passive networks containing resistance, capacitance and self-inductance but no transformers. As is well known, the theory of these networks may be based upon their nodal admittance matrices, submatrices and their determinants. In Sects. 2–4, this matrix algebra is developed. For the cofactor Δ_{12} which occurs in the various transfer functions, the known non-negativity of its coefficients is sharpened in Sect. 5. A number of examples involving surplus factors occur in Sect. 6. In particular, it is shown that triplets of admittance functions Y_{11} , Y_{12} , Y_{22} exist which can be synthesized as grounded RC two-ports only after the introduction of surplus factors. The last section examines the possibility of negative coefficients in $\Delta_{12}/(s + a)$ when $s + a$ is a surplus factor of Y_{11} , Y_{12} , Y_{22} and derives characteristic properties of the cofactor Δ_{1122} for this occurrence. This result indicates the existence either of additional realizability conditions or of a $\Delta_{12}/(s + a)$, ($s + a$ a common factor as above), containing some negative coefficients.

2. Cofactors of the network determinant. Consider a general RLC transformerless network and analyze it on a nodal basis. The nodes are identified so that each branch consists of an R, L and C in parallel. Hence the admittance y_{ij} ($i \neq j$) of the branch between nodes i and j is of the form $a + bs + c/s$ where $a \geq 0$, $b \geq 0$, $c \geq 0$, and, of course, $y_{ij} = y_{ji}$. We also write**

$$y_{ii} = \sum_{\substack{j=0\\ j\neq i}}^{t} y_{ij} \, , \tag{2.1}$$

where $t + 1$ is the total number of nodes. We introduce the notation

$$(ii) = y_{ii} \, , \qquad (ij) = -y_{ij}(i \neq j) \tag{2.2}$$

and write (2.1) as

$$\sum_{j=0}^{t} (ij) = 0. \tag{2.3}$$

Let I_i be the current impressed by the driving sources upon node i and let E_i be the voltage from any fixed node taken as reference node to node i. Then, using Kirchoff's laws, the equations of the nodal system may be written as

$$I_i = \sum_{j=0}^{t} (ij)E_j \tag{2.4}$$

*Received July 5, 1963; revised manuscript received Aug. 5, 1963. The results presented here were obtained in the course of research sponsored by the U. S. Army Research Office (Durham) under Grant DA-ARO (D)-31-124-G273.

**Throughout this paper indices have the following ranges: h, i, j, k, l: $0, 1, 2, \cdots, t$; p, q, r, u, v: $1, 2, \cdots, t$; w: $2, 3, \cdots, t$; $\alpha, \beta, \gamma, \delta, \zeta$: $3, 4, \cdots, t$.

The ranges of other indices are indicated when they appear. Each equation such as (2.1) is valid for all values in the range of the free index; that is, for $i = 0, 1, 2, \cdots, t$.

after reference to (2.1) and (2.2). These equations are not independent since the sum of the right members as well as of the left members of all $t + 1$ equations is zero. Furthermore, one of the voltages E_i does not actually appear in (2.4), since if node k is the reference node then $E_k \equiv 0$.

The network determinant D is the coefficient determinant in (2.4) $D = |(ij)|$. We also write D_{ij} for the cofactor of (ij) in D, D_{ijkl} for the cofactor of (kl) in D_{ij}, and use an analogous notation for further iterated cofactors of D. It follows from (2.3) that $D = 0$. Also, writing Δ for D_{00},

$$\Delta = |(pq)|. \tag{2.5}$$

If the last $t - 1$ columns of Δ are added to the first column and use is made of (2.3), the transformed determinant is D_{01}. Consequently $D_{00} = D_{01}$. Interchanging the roles of nodes 0, 1, we obtain $D_{11} = D_{10}$. Hence, using the symmetry of the y_{ij} in i, j, $\Delta = D_{00} = D_{10} = D_{11}$. Similarly $D_{00} = D_{po} = D_{pp} = D_{pq} = D_{qq}$. This proves that

$$\Delta = D_{ij}. \tag{2.6}$$

Relative to the reference node k, the open circuit input impedance Z_{ii} (between nodes k and i) and transfer impedance Z_{ij} are defined by

$$Z_{ii} = \frac{E_i}{I_i}, \qquad Z_{ij} = Z_{ji} = \frac{E_j}{I_i}$$

subject to the constraint that all independent driving currents other than I_i are zero. Similarly the short circuit admittance ${}_{(j)}Y_{ii}$ (relative to node j) and transfer admittance Y_{ij} are defined by

$${}_{(j)}Y_{ii} = \frac{I_i}{E_i}, \qquad Y_{ij} = Y_{ji} = \frac{I_j}{E_i}$$

subject to the above constraint and also $E_j = 0$. It follows readily from these definitions and the network equations (2.4) that

$$\begin{aligned} Z_{ii} &= \frac{D_{kkii}}{D_{kk}}, \qquad Z_{ij} = Z_{ji} = \frac{D_{kkij}}{D_{kk}}, \\ {}_{(j)}Y_{ii} &= \frac{D_{kkjj}}{D_{kkiijj}}, \qquad Y_{ij} = Y_{ji} = -\frac{D_{kkij}}{D_{kkiijj}}. \end{aligned} \tag{2.7}$$

Here k is the reference node and I_k is the dependent current (equal to $-I_i$ in this case). If node k is taken to be node 0, since $\Delta \equiv D_{00}$, (2.7) yields the familiar equations

$$\begin{aligned} Z_{pp} &= \frac{\Delta_{pp}}{\Delta} \qquad Z_{pq} = Z_{qp} = \frac{\Delta_{pq}}{\Delta}, \\ {}_{(q)}Y_{pp} &= \frac{\Delta_{qq}}{\Delta_{ppqq}} \qquad Y_{pq} = Y_{qp} = -\frac{\Delta_{pq}}{\Delta_{ppqq}}. \end{aligned} \tag{2.8}$$

Examination of the structure of Δ_{pp} proves that it is the Δ of the network after identification or short circuiting of nodes o and p (Analytically, let $y_{po} = \infty$). Similarly Δ_{ppqq} is the Δ of the network after identification of nodes o, p, q, and analogous results hold for higher iterated cofactors. These remarks are verified by (2.8) which shows that if nodes o and q are short circuited then the resulting Z_{pp} is the reciprocal of ${}_{(q)}Y_{pp}$, in accordance with their definitions.

The iterated cofactors of D are not independent. Thus

$$D_{0012} + D_{1102} = \begin{vmatrix} -(21) & (23) & \cdots & (2t) \\ -(31) & (33) & \cdots & (3t) \\ & \cdots & & \\ -(t1) & (t3) & \cdots & (tt) \end{vmatrix} + \begin{vmatrix} -(20) & (23) & \cdots & (2t) \\ -(30) & (33) & \cdots & (3t) \\ & \cdots & & \\ -(t0) & (t3) & \cdots & (tt) \end{vmatrix}$$

$$= \begin{vmatrix} -[(21) + (20)] & (23) & \cdots & (2t) \\ -[(31) + (30)] & (33) & \cdots & (3t) \\ & \cdots & & \\ -[(t1) + (t0)] & (t3) & \cdots & (tt) \end{vmatrix}$$

If all the columns except the first are subtracted from the first column and use is made of (2.3), we obtain $D_{0011} = D_{0012} + D_{1102}$ or

$$\Delta_{11} = \Delta_{12} + D_{1102} \, . \tag{2.9}$$

More generally,

$$D_{iijj} = D_{iijk} + D_{jjik} \, . \tag{2.10}$$

Of course, $D_{iijk} = D_{iikj}$. As a simple application of (2.10), we note that

$$D_{oopp} + D_{ooqq} - 2D_{oopq} = \Delta_{pp} + \Delta_{qq} - 2\Delta_{pq} = D_{ppqq} \, . \tag{2.11}$$

It follows from (2.10) and (2.7) that $Z_{pp} + Z_{qq} - 2Z_{pq}$ equals the input impedance between nodes p and q. There is a similar interpretation for $Y_{pp} + Y_{qq} + 2Y_{pq}$.

The proof of (2.10) may be generalized to apply to higher iterated cofactors of D. In this way, we establish

$$D_{iijjkk} = D_{iijjkl} + D_{jjkkil} + D_{kkiijl} \tag{2.12}$$

and analogous results for higher iterated cofactors.

Finally, we note a result which is true for the iterated cofactors of any determinant and does not depend upon the special structure of D. Let $M_{ijkl} \cdots$ be the minor obtained by deleting rows $i, k, \cdots$ and columns $j, l, \cdots$ from D and make the unessential assumption $k > i$. Then, by definition, $D_{ij} = (-1)^{i+j} M_{ij}$, $D_{ijkl} = (-1)^{i+j}(-l)^{(k-l)+l_1} . M_{ijkl}$, where $l_1 = l - 1$ if $l > j$ and $l_1 = l$ if $l < j$. Similarly $D_{kj} = (-l)^{k+j} M_{kj}$, $D_{kjil} = (-1)^{k+j}(-1)^{i+l_1} M_{ijkl}$. Comparison of the above equations leads to

$$D_{ijkl} = -D_{kjil} \, . \tag{2.13}$$

Similarly, for an interchange of two column indices,

$$D_{ijkl} = -D_{ilkj} \, . \tag{2.14}$$

Thus, in an iterated cofactor of any determinant, the interchange of two row (or column) indices changes the sign of the cofactor.

3. The network matrix. The network admittance matrix $\| (ij) \|$ has rank less than $t + 1$, since its determinant D is zero. In consequence of (2.6), its rank equals that of the admittance matrix Y defined by

$$Y = \|(pq)\|. \tag{3.1}$$

If G, C, Γ denote the corresponding matrices of conductance, capacitance, and inverse inductance respectively then the admittance matrix may be written

$$Y = G + sC + \frac{1}{s}\,\Gamma. \tag{3.2}$$

Let us choose any one of the matrices on the right side of (3.2), say C, and consider the quadratic form

$$F = \sum_{p=1}^{t} C_{pp}x_p^2 - \sum_{\substack{p,q=1\\ p\neq q}}^{t} C_{pq}x_px_q\,,$$

whose matrix is C. According to (2.1), F may be written as

$$F = \sum_{p=1}^{t} C_{po}x_p^2 + \sum_{q>p} C_{pq}(x_p - x_q)^2. \tag{3.3}$$

Since the C_{po}, C_{pq} are all non-negative, it follows that F is a positive semi-definite form. Hence C *is a positive semi-definite matrix*. The same result* also holds for the matrices G and Γ.

Let us now consider the rank of C. The indices $1, 2, \cdots, t$ may be separated into sets $(p_1, p_2, \cdots, p_{t_1})$, $(q_1, q_2, \cdots, q_{t_2})$, $\cdots$ in accordance with the following rules:

(i) For any indices p, q belonging to different sets, $C_{pq} = 0$;

(ii) For any pair of indices p_1, p_2 belonging to the same set, either $C_{p_1p_2} \neq 0$ or there exists a chain of indices $p_1, p_a, p_b, \cdots, p_f, p_2$ such that $C_{p_1p_a} \neq 0$, $C_{p_ap_b} \neq 0, \cdots, C_{p_fp_2} \neq 0$.

Let C_p be the submatrix of C having rows and columns $p_1, p_2, \cdots, p_{t_1}$ and let F_p be the corresponding quadratic form. Here C_p is the capacitance matrix of a network N_p comprising nodes $p_1, p_2, \cdots, p_{t_1}$ and 0, where according to (ii), $p_1, p_2, \cdots p_{t_1}$ constitute a capacitance connected network. Also F_p is that part of F which contains the variables $x_{p_1}, x_{p_2}, \cdots, x_{p_{t_1}}$ only. Now the rank of C_p is equal to the number of independent variables in F_p. Also, from (ii), the number of independent variables $(x_{p_a} - x_{p_b})$ among the terms like $C_{p_1p_2}(x_{p_1} - x_{p_2})^2$ is $t_1 - 1$ since all these terms of F_p equal zero if and only if $x_{p_1} = x_{p_2} = \cdots = x_{p_{t_1}}$. If F_p also contains at least one term $C_{po}x_p^2$, then x_p is an additional independent variable. Hence the rank of C_p equals t_1 or $t_1 - 1$, according as the reference node 0 is capacitance connected or not to the network of the remaining nodes $p_1, p_2, \cdots, p_{t_1}$.

Since, in accordance with (i), the C_{pq} are all zero, the rank of C is equal to the sum of the ranks of the n constituent submatrices C_p, C_q, $\cdots$. The corresponding n networks N_p, N_q, $\cdots$ are not capacitance connected with each other except, possibly, through the common reference node 0. This proves the result that the rank of C equals $t + 1 - n$. A similar conclusion obtains for G and Γ. More generally, *the rank of the network admittance matrix equals the difference between the total number of nodes in the network and the minimum number of connected subnetworks composing it.* In particular, *the rank of the admittance matrix of a* $(t + 1)$-*node network is* t *if and only if the network is connected.*

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*This familiar theorem is usually not proved but simply assumed after reference to extraneous "energy" considerations. It is proved rigorously in [4, pp. 352–354], using ideas from linear graph theory. That $\| Z_{pq} \|$ and $\| Y_{pq} \|$ are positive real matrices follows readily from this result [4, pp. 355–356, 377–379].

4. Expansions of the Δ_{pq}. The determinant Δ_{12} is given by

$$\Delta_{12} = (-1) \begin{vmatrix} -y_{21} & -y_{23} & \cdots & -y_{2t} \\ -y_{31} & y_{33} & \cdots & -y_{3t} \\ & \cdots & & \\ -y_{t1} & -y_{t3} & \cdots & y_{tt} \end{vmatrix} \tag{4.1}$$

where all diagonal elements except y_{21} are defined by (2.1). Expanding (4.1) according to elements of the first column, we obtain

$$\Delta_{12} = \sum_{w=2}^{t} (-y_{w1})\Delta_{12w1}$$

or, using (2.14),

$$\Delta_{12} = \sum_{w=2}^{t} y_{w1}\Delta_{11w2}. \tag{4.2}$$

Now each $\Delta_{11\alpha 2}$ ($\alpha > 2$) may be expanded according to the elements of its first row. This gives

$$\Delta_{11\alpha 2} = \sum_{\beta=3}^{t} (-y_{2\beta})\Delta_{11\alpha 22\beta}$$

or, using (2.13),

$$\Delta_{11\alpha 2} = \sum_{\beta=3}^{t} y_{2\beta}\Delta_{1122\alpha\beta} .$$

Substitution in (4.2) yields the Cauchy expansion

$$\Delta_{12} = y_{21}\Delta_{1122} + \sum_{\alpha,\beta=3}^{t} y_{\alpha 1}y_{2\beta}\Delta_{1122\alpha\beta} . \tag{4.3}$$

In a similar manner, we may derive

$$\begin{aligned} \Delta_{11} &= y_{22}\Delta_{1122} - \sum_{\alpha,\beta=3}^{t} y_{\alpha 2}y_{2\beta}\Delta_{1122\alpha\beta} , \\ \Delta_{22} &= y_{11}\Delta_{1122} - \sum_{\alpha,\beta=3}^{t} y_{\alpha 1}y_{1\beta}\Delta_{1122\alpha\beta} . \end{aligned} \tag{4.4}$$

If (4.3) and (4.4) are divided by Δ_{1122} and use is made of (2.8), these equations may be written as

$$\begin{aligned} -Y_{12} &= y_{21} + \sum_{\alpha,\beta=3}^{t} y_{\alpha 1}y_{2\beta\,(12)}Z_{\alpha\beta} , \\ Y_{11} &= y_{22} - \sum_{\alpha,\beta=3}^{t} y_{\alpha 2}y_{2\beta\,(12)}Z_{\alpha\beta} , \\ Y_{22} &= y_{11} - \sum_{\alpha,\beta=3}^{t} y_{\alpha 1}y_{1\beta\,(12)}Z_{\alpha\beta} , \end{aligned} \tag{4.5}$$

where ${}_{(12)}Z_{\alpha\beta}$ is the value of $Z_{\alpha\beta}$ when nodes 1 and 2 are short circuited to node 0.

5. The coefficient relations. Every iterated cofactor of the form $D_{hhii\cdots jjkl}$ ($k \neq l$) may also be written similar to (4.1) by bringing the element $-y_{kl}$ to the upper left hand corner and consequently has those properties of Δ_{12} which only depend upon its expression (4.1). Now all the $\Delta_{11\alpha 2}$ $(\alpha > 2)$ in (4.2) have the same structure as Δ_{12}. Also, since Δ_{11} is the value of Δ after identification of nodes 0 and 1, it follows from (2.9) that Δ_{1122} may be written as a sum of determinants like Δ_{12}. (Otherwise, this last follows directly from (2.12)). Hence (4.2) shows that Δ_{12} may be written as a sum of products of similar determinants of lower order and the y_{w1}. Repeating this process for each determinant of lower order until order one is attained, we prove the following coefficient condition first stated in [1, pp. 124–125]: *Δ_{12} is a multilinear form in the y_{ij} $(i \neq j)$ with non-negative coefficients.* It follows at once from this result and (2.9) that *Δ_{11} and $\Delta_{11} - \Delta_{12}$ are each multilinear forms in the y_{ij} $(i \neq j)$ with non-negative coefficients.*

These results are true even if y_{ij} and y_{ji} are unequal. However, in the present case, $y_{ij} = y_{ji} = a + bs + c/s$, $a \geq 0$, $b \geq 0$, $c \geq 0$. Hence the preceding results may be stated in the form [1, 2]: *For an RLC network* without transformers, $s^{t-1}\Delta_{12}$ and $s^{t-1}(\Delta_{11} - \Delta_{12})$ are each polynomials in s with non-negative coefficients.*

We now investigate whether any of the coefficients in these polynomials may actually be zero. Let N be the given RLC network having the structure shown in Fig. 1. Here

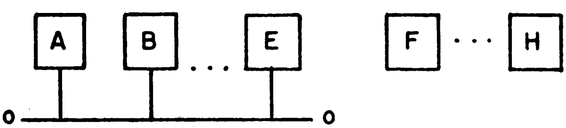

FIG. 1.

node 0 is shown explicitly and each block represents an internally connected network. Write ${}_0N$ for the network with node 0 deleted and all the y_{p0} set equal to zero and write ${}_0\Delta$, ${}_0\Delta_{pq}$ for the values of Δ, Δ_{pq} respectively when all the $y_{r0} = 0$. We first prove

Lemma 5.1. *If ${}_0N$ is a connected network, then for any nodes** p, q, u, v in it, $s^{t-1}\Delta_{pq}$ and $s^{t-1}\Delta_{uv}$ have a power of s in common.*

We note that ${}_0\Delta$ is the network determinant D of ${}_0N$ so that (2.6) holds. Hence $s^{t-1}\,{}_0\Delta_{pq} = s^{t-1}\,{}_0\Delta_{uv} = M$, where M is a polynomial in s with non-negative coefficients. According to Sect. 3, the rank of the network admittance matrix of ${}_0N$ equals $t - 1$ so that $M \not\equiv 0$. Now Δ_{pq}, Δ_{uv} may be written

$$s^{t-1}\Delta_{pq} = s^{t-1}\,{}_0\Delta_{pq} + f_{pq} = M + f_{pq}$$

$$s^{t-1}\Delta_{uv} = s^{t-1}\,{}_0\Delta_{uv} + f_{uv} = M + f_{uv}$$

where f_{pq}, f_{uv} are polynomials with non-negative coefficients each term of which contains at least one y_{r0} as a factor. Hence $s^{t-1}\Delta_{pq}$ and $s^{t-1}\Delta_{uv}$ have the polynomial M in common.

We now prove

Lemma 5.2. *Let node v not be connected to node u and let nodes† p, q be connected to node u in ${}_0N$. Then Δ_{uv} must be zero and $s^{t-1}\Delta_{up}$, $s^{t-1}\Delta_{uq}$ are zero or have a power of s in common.*

*If the network is an RC network, the factor s^{t-1} may be deleted.

**These nodes need not be distinct.

†The nodes p, q need not be distinct.

The determinant Δ may be partitioned in accordance with the separate blocks $A, B, \cdots, H$ of N. Then Δ has the structure

$$\begin{vmatrix} \Delta_A & & & & O \\ & \Delta_B & & & \\ & & \Delta_C & & \\ & & & \ddots & \\ O & & & & \Delta_H \end{vmatrix}$$

where $\Delta_A, \cdots, \Delta_H$ are the Δ's of the blocks $A, \cdots, H$ each augmented by node 0. Clearly, if u and v are chosen from different blocks, $\Delta_{uv} = 0$, proving the first part of the lemma. We now consider various possibilities, noting that in Fig. 1 blocks $A \cdots E$ are specifically connected to datum node 0, and blocks $F \cdots H$ are not.

(i) N consists of $A \cdots E, F$ and at least one more connected network. Then, according to Sect. 3, the rank of the network admittance matrix is less than $t - 1$ so that all the $\Delta_{rp} = 0$.

(ii) N consists of $A \cdots E$ and F. Then if u, p are nodes in A, $\Delta_{up} = \Delta_{Aup}\Delta_B \cdots \Delta_E\Delta_F$. Since F is not connected to 0, $\Delta_F = 0$ so that $\Delta_{up} = 0$. Similar conclusions hold when u, p are both in $B, \cdots, E$. Suppose u, p are nodes in F. Then $\Delta_{up} = \Delta_A\Delta_B \cdots \Delta_E\Delta_{Fup}$. Here Δ_F has the structure of D so that (2.6) holds. Consequently if q is any other node in F, $\Delta_{Fup} = \Delta_{Fuq}$, so that $\Delta_{up} = \Delta_{uq}$ in this case.

(iii) N consists of a connected network $A \cdots E$. If u, p, q are nodes in A, $\Delta_{up} = \Delta_{Aup}\Delta_B \cdots \Delta_E$ and $\Delta_{uq} = \Delta_{Auq}\Delta_B \cdots \Delta_E$. According to Lemma 5.1, after multiplication by a suitable power of s, Δ_{Aup} and Δ_{Auq} have some power of s in common. Consequently the same must be true of $s^{t-1}\Delta_{up}$ and $s^{t-1}\Delta_{uq}$. A similar proof applies to nodes in each of the remaining blocks* $B, \cdots, E$. This completes the proof of the lemma.

We now prove the coefficient relations stated in the following theorems.

Theorem 5.1. *In any RC network, Δ_{pq} and $\Delta_{pp} - \Delta_{pq}$ are polynomials in s without any missing powers of s, all of whose coefficients are positive.*

Theorem 5.2. *In any $(t + 1)$-node RLC network without transformers, $s^{t-1}\Delta_{pq}$ and $s^{t-1}(\Delta_{pp} - \Delta_{pq})$ are polynomials in s with non-negative coefficients. Power gaps may occur but the coefficients of two consecutive powers of s cannot be zero unless all subsequent coefficients vanish.*

The proof of Theorem 5.1 follows; the proof for Theorem 5.2 is completely analogous. We begin by noting that if $t = 2$, $\Delta_{11} = y_{22}$ and $\Delta_{12} = y_{21}$. Hence Theorem 5.1 is true in this case. Assume the theorem is true for all networks with $t < \tau$ and let Δ belong to any RC network N for which $t = \tau$. Let N' be the network (with $t = \tau - 1$) formed by the short circuit of nodes 0 and 1 and let its corresponding Δ be Δ'. Then $\Delta' = \Delta_{11}$, as indicated in Sect. 2. From (4.2),

$$\Delta_{12} = \sum_{\sigma=2}^{\tau} y_{\sigma 1}\Delta'_{\sigma 2}$$

*However, if u, p are in A and v, q are in B, then Δ_{up} and Δ_{vq} need not have some power of s in common.

By the hypothesis of the induction, the theorem is true for each $\Delta'_{\sigma 2}$ and hence also for the products $y_{\sigma 1}\Delta'_{\sigma 2}$. Now for all nodes σ not connected to node 2 in ${}_oN'$, $\Delta'_{\sigma 2} = 0$, so that no contribution to Δ_{12} is made by such $\Delta'_{\sigma 2}$. Furthermore, for all nodes σ connected to node 2 in ${}_oN'$, the corresponding $\Delta'_{\sigma 2}$ have a power of s in common, according to Lemma 5.2. Consequently the corresponding products $y_{\sigma 1}\Delta'_{\sigma 2}$ either have a power of s in common or have terms containing consecutive powers of s. Hence the sum of all these products constitute a polynomial in s with positive coefficients and no missing powers. A similar conclusion may be proved for any Δ_{pq} belonging to N. This completes the induction for the Δ_{pq} and proves Theorem 5.1 for this case. A similar result for $\Delta_{pp} - \Delta_{pq}$ follows immediately from (2.9).

6. Some illustrative examples. The preceding section has shown that for an RC network, Δ and its cofactors have no power gaps and positive coefficients. However if the ratios of these determinants are considered, it is well known that even negative coefficients can occur after deletion of possible common factors. The earliest example appears to be that of a voltage transfer function Δ_{12}/Δ_{11} whose numerator has a negative coefficient [1, pp. 120–122]. An example of a 5 node network whose transfer impedance Z_{12} has a negative coefficient appears in [5, p. 92]. In this latter paper, Slepian and Weinberg prove that if Δ_{11}, Δ_{12} and Δ_{1122} all have a common factor* $s + a$ and if $t \leq 4$, then $\Delta_{12}/(s + a)$ has all non-negative coefficients. The authors speculate concerning the truth of this theorem if $t > 4$. The following counter example shows that the conclusion of this theorem may be false if $t > 4$.

$$\Delta = \begin{vmatrix} 1 & 0 & -1 & 0 & 0 \\ 0 & s+1 & 0 & -1 & -s \\ -1 & 0 & 2(s+1) & -1 & -s \\ 0 & -1 & -1 & 4s+3 & 0 \\ 0 & -s & -s & 0 & 3s+4 \end{vmatrix}.$$

For the corresponding 6 node RC network, Δ, Δ_{12}, Δ_{1122} each has a simple zero at $s = -1$, Δ_{11} has a double zero and $\Delta_{22} \neq 0$. We find that

$$\Delta_{12} = (s + 1)(4s^2 - s + 4)$$

so that both Z_{12} and $-Y_{12}$ have a negative coefficient after common factor deletion.

In all the preceding examples, a common factor has been present in some but not all of the determinants that arise in RC two-port synthesis. In the synthesis problem, the triplet of admittance functions Y_{11}, Y_{12}, Y_{22} (or Z_{11}, Z_{12}, Z_{22}) completely specifying the external behavior of the two-port are given and the network must be determined. This specification determines all the network determinants

$$\Delta,\ \Delta_{11},\ \Delta_{12},\ \Delta_{22},\ \Delta_{1122} \tag{6.1}$$

up to a common multiplicative factor. Under these circumstances, is it possible for $\Delta_{12}/(s + a)$ to have a zero or negative coefficient after deletion of a factor $(s + a)$ present in each of the determinants (6.1)? Stated otherwise, do triplets Y_{11}, Y_{12}, Y_{22} exist which cannot be realized as grounded RC two-ports except by the introduction of a common factor into all the determinants (6.1)?

*Here $a \geq 0$, since all the eigenvalues of Δ or any principal minor must be non-negative according to classical algebraic theory.

A partial answer is afforded by the following example which proves that $\Delta_{12}/(s + a)$ may have missing powers. Let

$$\begin{aligned} Y_{11} = Y_{22} &= \frac{s}{2} + \frac{1}{2} + \frac{s}{s+1} = \frac{s^2 + 4s + 1}{2(s+1)}, \\ -Y_{12} &= \frac{s}{2} + \frac{1}{2} - \frac{s}{s+1} = \frac{s^2 + 1}{2(s+1)}. \end{aligned} \tag{6.2}$$

These admittance functions may be realized by the grounded RC two-port whose Δ is given by

$$\Delta = \begin{vmatrix} s+1 & 0 & -s & -1 \\ 0 & s+1 & -s & -1 \\ -s & -s & 2s+2 & 0 \\ -1 & -1 & 0 & 2s+2 \end{vmatrix}.$$

Here $\Delta_{11} = \Delta_{22} = 2(s + 1)(s^2 + 4s + 1)$, $\Delta_{12} = 2(s + 1)(s^2 + 1)$, $\Delta_{1122} = 4(s + 1)^2$, $\Delta = 8s(s + 1)^2$. According to Theorem 5.1, the functions (6.2) cannot be realized except by the introduction of a surplus factor into all the network determinants (6.1). It is the first example of a realization problem in which the behavior of an RC two-port is completely specified and can only be synthesized in this manner.

7. The structure of $\Delta_{12}/(s + a)$. Examples similar to (6.2) may be constructed in which the numerator of $-Y_{12}$ has a negative coefficient. These admittance triplets satisfy all known realizability conditions for a grounded RC two-port but their synthesis has not been accomplished thus far. These circumstances raise the question of whether $\Delta_{12}/(s + a)$ may have a negative coefficient when $(s + a)$, $a > 0$, is a common factor of all the determinants (6.1). We now consider this problem. The reader is reminded that the ranges of the various subscripts which appear in the sequel are indicated in the footnote to equation (2.1).

We assume that each of (6.1) for an RC network has the factor $s + a$. If $f(s)$ is divisible by $s + a$, we write $f(s) \simeq 0$. Now by a theorem of Jacobi

$$\Delta_{pq}\Delta_{uv} - \Delta_{pv}\Delta_{uq} = \Delta\Delta_{pquv} \tag{7.1}$$

is true for any arbitrary determinant Δ. Since the network determinant Δ is symmetric, it follows from (7.1) with $p = q = 1$, $u = v$ that $\Delta_{1u} \simeq 0$. Similarly $\Delta_{2u} \simeq 0$. If Δ is replaced by Δ_{1r}, then (7.1) with $p = q = 2$, $u = v$ proves that $\Delta_{1r2u} \simeq 0$. Consequently

$$\Delta \simeq 0, \qquad \Delta_{1u} \simeq 0, \qquad \Delta_{2u} \simeq 0, \qquad \Delta_{1r2u} \simeq 0 \tag{7.2}$$

This means that the $(t - 2) \times t$ matrix

$$[(\alpha p)_o] \tag{7.3}$$

has rank less than $(t - 2)$. Here $(\alpha p)_o$ is the value of (αp), defined by (2.2), when $s = -a$. It follows that the row vectors of (7.3) are linearly dependent; that is, constants K_h, not all zero, exist so that

$$\sum_{\alpha=3}^{t} K_\alpha(\alpha p)_o = 0. \tag{7.4}$$

Conversely, equations (7.4) are sufficient conditions for the existence of a common factor $s + a$ as stated in (7.2).

If some K_α, say K_3, equals zero, then $\Delta_{112233} \simeq 0$ since all minors of (7.3) containing rows $4, 5, \cdots, t$ are zero. Similarly, using the Laplace expansion of Δ_{1133} or Δ_{2233} according to elements of the top row, we find that $\Delta_{1133} \simeq 0$, $\Delta_{2233} \simeq 0$. To simplify the subsequent discussion, we assume that none of the $\Delta_{1122\alpha\alpha}$ are equal to zero. This assumption implies that $K_3K_4 \cdots K_t \neq 0$. This restriction is not essential, as the excluded cases may be treated by a modification of the method employed in the sequel.

We introduce the notation

$$d_{\alpha\beta} = \Delta_{1122\alpha\beta}(s), \qquad d_{\alpha\beta 0} = d_{\alpha\beta}(-a). \tag{7.5}$$

It follows from the last $t - 2$ equations of (7.4) that

$$\frac{d_{\gamma\alpha 0}}{d_{\gamma\beta 0}} = \frac{K_\alpha}{K_\beta}. \tag{7.6}$$

Let $c = d_{330}/|d_{330}|$ and choose K_3 in (7.4) so that $d_{330} = cK_3^2$. It follows from (7.6) that

$$d_{\alpha\beta 0} = cK_\alpha K_\beta . \tag{7.7}$$

Now according to classical eigenvalue theory, the roots of each $d_{\alpha\alpha}(s)$ separate those of $\Delta_{1122}(s)$. Also $d_{\alpha\alpha}(0) \geq 0$ and $d_{\alpha\alpha}(-a) \neq 0$. Hence all the $d_{\alpha\alpha}$ have the same sign at $s = -a$. Consequently the K_α determined by (7.7) with $\alpha = \beta$ are all real.

We now consider the structure of $\Delta_{12}/(s + a)$. Without loss of generality we may choose $a = 1$. Any Δ_{12} may be constructed from its minor Δ_{1122} by bordering Δ_{1122} with a row and column of elements in accordance with (4.1). With this in mind, we investigate whether a Δ_{12} may be generated from a given Δ_{1122}, with $\Delta_{1122}(-1) = 0$, by adding an arbitrary realizable border of elements satisfying (7.4) such that $\Delta_{12}/(s + 1)$ has some negative coefficients.

Let

$$y_{pq} = a_{pq}s + b_{pq} ,$$

$$d_{\alpha\beta} = \sum_{\nu=0}^{m} e_{\alpha\beta}^{\nu} s^{\nu},$$

$$\Delta_{1122} = \sum_{\sigma=0}^{m+1} \epsilon^{\sigma} s^{\sigma},$$

$$\Delta_{12} = \sum_{\tau=0}^{m+2} A_\tau s^\tau,$$

$$\Delta_{12}/(s + 1) = \sum_{\sigma=0}^{m+1} B_\sigma s^\sigma,$$

where, in general, $m = t - 3$. Now according to (4.3) and (7.5),

$$\Delta_{12} = y_{21}\Delta_{1122} + \sum_{\alpha,\beta=3}^{t} y_{\alpha 1} y_{2\beta} d_{\alpha\beta} .$$

Substitution of the preceding equations into the last equation yields

$$A_\tau = a_{21}\epsilon^{\tau-1} + b_{21}\epsilon^\tau + \sum_{\alpha,\beta} [b_{\alpha 1}b_{2\beta}e_{\alpha\beta}^{\tau} + (a_{\alpha 1}b_{2\beta} + b_{\alpha 1}a_{2\beta})e_{\alpha\beta}^{\tau-1} + a_{\alpha 1}a_{2\beta}e_{\alpha\beta}^{\tau-2}],$$

$$\tau = 0, 1, 2, \cdots, m + 2, \tag{7.8}$$

$$B_\sigma = A_\sigma - B_{\sigma-1}, \qquad \sigma = 0, 1, 2, \cdots, m+1, \tag{7.9}$$

where $\epsilon^{-\nu} = \epsilon^{m+1+\nu} = e_{\alpha\beta}^{-\nu} = e_{\alpha\beta}^{m+\nu} = 0$ if $\nu > 0$. From (7.8) and (7.9), we obtain

$$B_\sigma = b_{21}C^\sigma + a_{21}C^{\sigma-1} + \sum_{\alpha,\beta} [b_{\alpha 1}b_{2\beta}C_{\alpha\beta}^\sigma + (a_{\alpha 1}b_{2\beta} + b_{\alpha 1}a_{2\beta})C_{\alpha\beta}^{\sigma-1} + a_{\alpha 1}a_{2\beta}C_{\alpha\beta}^{\sigma-2}],$$

$$\sigma = 0, 1, 2, \cdots, m+1, \tag{7.10}$$

where

$$C^\sigma = \epsilon^\sigma - \epsilon^{\sigma-1} + \epsilon^{\sigma-2} - \cdots \pm \epsilon^o = \epsilon^\sigma - C^{\sigma-1},$$

$$C_{\alpha\beta}^\sigma = e_{\alpha\beta}^\sigma - e_{\alpha\beta}^{\sigma-1} + e_{\alpha\beta}^{\sigma-2} - \cdots \pm e_{\alpha\beta}^o = e_{\alpha\beta}^\sigma - C_{\alpha\beta}^{\sigma-1},$$

so that

$$\Delta_{1122}/(s+1) = \sum_{\sigma=0}^{m} C^\sigma s^\sigma, \tag{7.11}$$

$$d_{\alpha\beta}/(s+1) = \sum_{\sigma=0}^{m-1} C_{\alpha\beta}^\sigma s^\sigma + \frac{C_{\alpha\beta}^m s^m}{s+1}. \tag{7.12}$$

Note also that $C^{-\nu} = C_{\alpha\beta}^{-\nu} = 0$, $C^{m+1+\nu} = -C^{m+\nu}$, $C_{\alpha\beta}^{m+\nu} = -C_{\alpha\beta}^{m+\nu-1}$ for $\nu > 0$. Since Δ_{1122} is divisible by $(s+1)$, we readily find that $B_o \geq 0$ and $B_{m+1} \geq 0$.

We now investigate the conditions which must be satisfied by Δ_{1122} if the remaining coefficients $B_1, B_2, \cdots, B_m$ are to be non-negative for all realizable borders of Δ_{1122} satisfying (7.4). While a solution may be obtained by other means, it is convenient to phrase the problem as one in linear programming. Write B_σ, given by (7.10), in the form

$$B_\sigma = \sum_{w=2}^{t} (F_w^\sigma a_{w1} + G_w^\sigma b_{w1}), \qquad \sigma = 1, 2, \cdots, m. \tag{7.13}$$

The variables a_{w1}, b_{w1} are subject to the realizability inequalities

$$\begin{gathered} a_{w1} \geq 0, \qquad b_{w1} \geq 0, \\ -a_{\alpha 1} \geq -c_\alpha, \qquad -b_{\alpha 1} \geq -d_\alpha, \\ \sum_{\alpha=3}^{t} K_\alpha(a_{\alpha 1} - b_{\alpha 1}) \geq 0, \qquad \sum_{\alpha=3}^{t} -K_\alpha(a_{\alpha 1} - b_{\alpha 1}) \geq 0. \end{gathered} \tag{7.14}$$

Here

$$c_\alpha = a_{\alpha\alpha} - \sum_{\substack{\beta=3 \\ \beta\neq\alpha}}^{t} a_{\alpha\beta}, \qquad d_\alpha = b_{\alpha\alpha} - \sum_{\substack{\beta=3 \\ \beta\neq\alpha}}^{t} b_{\alpha\beta},$$

and the last inequalities are equivalent to (7.4) with $p = 1$.

Consider the values assumed by B_σ in the convex space S of the variables a_{w1}, b_{w1} satisfying (7.14). Since the origin lies in S, the value $B_\sigma = 0$ is assumed in S. Therefore the minimum value m of B_σ obeys $m \leq 0$. We determine the conditions that guarantee $m = 0$.

The linear programming problem and its dual problem are exhibited in the customary fashion [3, p. 55] by the matrix

$$\begin{array}{c|cccccccc|c}
 & x_1 & x_2 & x_3 & x_4 & \cdots & x_{2t-3} & x_{2t-2} & \leqq \\
a_{21} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & F_2^\sigma \\
b_{21} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & G_2^\sigma \\
a_{31} & K_3 & -K_3 & -1 & 0 & \cdots & 0 & 0 & F_3^\sigma \\
b_{31} & -K_3 & K_3 & 0 & -1 & \cdots & 0 & 0 & G_3^\sigma \\
\vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots \\
a_{t1} & K_t & -K_t & 0 & 0 & \cdots & -1 & 0 & F_t^\sigma \\
b_{t1} & -K_t & K_t & 0 & 0 & \cdots & 0 & -1 & G_t^\sigma \\
\geqq & 0 & 0 & -c_3 & -d_3 & \cdots & -c_t & -d_t &
\end{array}$$

According to the duality theorem [3, p. 60] the minimum m for the original problem equals the maximum M for the dual problem. Referring to the above matrix this dual problem is: Subject to the inequalities

$$\begin{gathered}
x_\tau \geq 0, \qquad \tau = 1, 2, \cdots, 2t-2, \\
0 \leq F_2^\sigma, \qquad 0 \leq G_2^\sigma, \qquad \sigma = 1, 2, \cdots, m, \\
-x_{2\alpha-3} + K_\alpha(x_1 - x_2) \leq F_\alpha^\sigma, \\
-x_{2\alpha-2} - K_\alpha(x_1 - x_2) \leq G_\alpha^\sigma,
\end{gathered} \tag{7.15}$$

find the maximum M of

$$B = -\sum_{\alpha=3}^{t} (c_\alpha x_{2\alpha-3} + d_\alpha x_{2\alpha-2}) + 0\cdot x_1 + 0\cdot x_2 .$$

For M to equal zero, since $c_\alpha \geq 0$, $d_\alpha \geq 0$, it is necessary and sufficient for all x_τ appearing in B to equal zero while the inequalities (7.15) remain true.

From (7.10), (7.11) and (7.13) $F_2^\sigma = C^{\sigma-1} \geq 0$ and $G_2^\sigma = C^\sigma \geq 0$, so these inequalities (7.15) are true. The remaining inequalities may be written $x_\tau \geq 0$ and

$$-G_2^\sigma - x_{2\alpha-2} \leq K_\alpha(x_1 - x_2) \leq F_\alpha^\sigma + x_{2\alpha-3} . \tag{7.16}$$

Assuming all c_α, d_α are not zero*, the condition $M = 0$ requires that the inequalities

$$-\frac{G_\alpha^\sigma}{K_\alpha} \leq (x_1 - x_2) \leq \frac{F_\alpha^\sigma}{K_\alpha}, \qquad K_\alpha > 0,$$

$$-\frac{F_\gamma^\sigma}{|K_\gamma|} \leq (x_1 - x_2) \leq \frac{G_\gamma^\sigma}{|K_\gamma|}, \qquad K_\gamma < 0$$

derived from (7.16) with $x_3 = x_4 = \cdots = x_{2t-2} = 0$, be consistent. Since x_1, x_2 may take on arbitrary non-negative values, these last inequalities are equivalent to

*If any $c_\alpha = 0$ or $d_\alpha = 0$, a continuity argument shows that the inequalities (7.17) must also hold in this case if $B_\sigma \geq 0$.

$$\begin{aligned}
&\frac{F_\alpha^\sigma}{K_\alpha} + \frac{G_\beta^\sigma}{K_\beta} \geq 0, \quad \text{if} \quad K_\alpha > 0, \qquad K_\beta > 0,\\
&\frac{F_\alpha^\sigma}{K_\alpha} + \frac{F_\gamma^\sigma}{|K_\gamma|} \geq 0, \quad \text{if} \quad K_\alpha > 0, \qquad K_\gamma < 0,\\
&\frac{G_\alpha^\sigma}{K_\alpha} + \frac{G_\gamma^\sigma}{|K_\gamma|} \geq 0, \quad \text{if} \quad K_\alpha > 0, \qquad K_\gamma < 0,\\
&\frac{F_\gamma^\sigma}{|K_\gamma|} + \frac{G_\delta^\sigma}{|K_\delta|} \geq 0, \quad \text{if} \quad K_\gamma < 0, \qquad K_\delta < 0,
\end{aligned} \tag{7.17}$$

where the indices α, β, γ, δ take on all values in the range $3, 4, \cdots, t$ for which $K_\alpha > 0$, $K_\beta > 0$, $K_\gamma < 0$, $K_\delta < 0$.

Reference to (7.10) and (7.13) shows that the left members of (7.17) are linear forms in the variables $a_{2\beta}$ and $b_{2\beta}$, $\beta = 3, 4, \cdots, t$. The procedure employed to minimize B_σ may now be followed for each of these linear forms. After writing each linear form similar to (7.13) to exhibit the coefficients of $a_{2\beta}$, $b_{2\beta}$ the result (7.17) is applied to it. In this way, a set of inequalities are obtained which, after elimination of duplications, may be summarized as follows:

Let α, γ be row indices and β, δ column indices of Δ_{1122} which determine four vertices $\alpha\beta$, $\alpha\delta$, $\gamma\beta$, $\gamma\delta$. Then

$$\frac{C_{\alpha\beta}^{\sigma-2}}{|C_{\alpha\beta}^m|} + \frac{C_{\gamma\beta}^{\sigma-1}}{|C_{\gamma\beta}^m|} + \frac{C_{\alpha\delta}^{\sigma-1}}{|C_{\alpha\delta}^m|} + \frac{C_{\gamma\delta}^{\sigma}}{|C_{\alpha\delta}^m|} \geq 0 \tag{7.18$_1$}$$

if the $C_{\nu\tau}^m$ at all four vertices have like signs,

$$\frac{C_{\alpha\beta}^{\sigma}}{|C_{\alpha\beta}^m|} + \frac{C_{\gamma\beta}^{\sigma}}{|C_{\gamma\beta}^m|} + \frac{C_{\alpha\delta}^{\sigma-1}}{|C_{\alpha\delta}^m|} + \frac{C_{\gamma\delta}^{\sigma-1}}{|C_{\gamma\delta}^m|} \geq 0 \tag{7.18$_2$}$$

if $C_{\alpha\beta}^m$, $C_{\alpha\delta}^m$ are of one sign, $C_{\gamma\beta}^m$, $C_{\gamma\delta}^m$ are of opposite sign, and

$$\frac{C_{\alpha\beta}^{\sigma}}{|C_{\alpha\beta}^m|} + \frac{C_{\gamma\beta}^{\sigma}}{|C_{\gamma\beta}^m|} + \frac{C_{\alpha\delta}^{\sigma}}{|C_{\alpha\delta}^m|} + \frac{C_{\gamma\delta}^{\sigma}}{|C_{\gamma\delta}^m|} \geq 0 \tag{7.18$_3$}$$

if $C_{\alpha\beta}^m$, $C_{\gamma\delta}^m$ are of one sign, $C_{\alpha\delta}^m$, $C_{\gamma\beta}^m$ are of opposite sign. In writing (7.18) in this form, we have used

$$C_{\alpha\beta}^m = (-1)^m c K_\alpha K_\beta ,$$

which follows from (7.7) and (7.12), to replace the products $K_\alpha K_\beta$. The $C_{\alpha\beta}^\sigma$ are defined by (7.12). This completes the proof of the theorem.

Theorem 7.1. *Let Δ_{1122} be any RC network determinant divisible by $s + 1$ none of whose principal minors is so divisible. Then, for any realizable Δ constructed by bordering Δ_{1122} with two rows and columns such that Δ, Δ_{11}, Δ_{12}, Δ_{22} are each divisible by $s + 1$, inequalities* (7.18) *are the necessary and sufficient conditions that all the coefficients of $\Delta_{12}/(s + 1)$ be non-negative.*

Three possibilities exist concerning conditions (7.18): (i) The conditions (7.18) do in fact apply to all Δ_{1122}'s referred to in Theorem 7.1 and are a consequence of known realizability conditions. (ii) These conditions apply to all these Δ_{1122}'s and constitute additional independent realizability conditions. (iii) These conditions do not apply to all these Δ_{1122}'s, leading to the construction of a counter example; that is, a grounded RC two-port for which the determinants (6.1) are all divisible by $s + 1$ and $\Delta_{12}/(s + 1)$ has some negative coefficients. Under (i) and (ii), the corresponding $\Delta_{12}/(s + 1)$ would have non-negative coefficients only. These possibilities will be investigated elsewhere.

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