

The Euler equations for (6) constitute the complete system of equations of an anisotropic plate. It is noted that the term $P_i^{*(n)}$ in the stress equation of motion given by (6) is a function of the surface traction. This automatically takes care of the traction boundary conditions in the interior of the plate, while the integral over A_n in (6) yields the displacement boundary conditions in the interior of the plate. In fact, (6) yields in a simple manner all conditions at the edge as well as in the interior of the plate for equations of all orders.

It is noted that the plate equations written from (6) are essentially those given by Mindlin [4, 5], who used the ordinary variational equation of motion in deriving the stress equations of motion and Neumann's method in deriving the boundary conditions. Mindlin's procedure is thus also justified from the variational point of view.

There is no difficulty in extending the present treatment of plates to shells. The results are reported elsewhere [6].

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A CORRESPONDENCE PRINCIPLE FOR VISCOELASTIC WAVE PROPAGATION*

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Introduction. It is a well known result of classical linear elasticity theory that the speed of propagation U of an acceleration wave, at a point x in an elastic solid of elasticity tensor $c_{ijkl}(x)$ and density $\rho(x)$, is a solution to the eigenvalue problem**

$$[c_{ijkl}(x)n_j n_l - \rho(x)U^2 \delta_{ik}]a_k = 0. \quad (1.1)$$

Here the unit vector n is the direction of propagation at x . Since the elasticity tensor obeys the symmetry relations

$$c_{ijkl} = c_{klij}, \quad (1.2)$$

it is clear that (1.1) has three (not necessarily distinct) solutions U_1, U_2, U_3 . More-

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**We use indicial notation and Cartesian tensors throughout. Thus subscripts have the range (1, 2, 3), x_i denote rectangular Cartesian coordinates, and $x = (x_1, x_2, x_3)$. Moreover summation over repeated subscripts is implied, δ_{ij} is Kronecker's delta, subscripts preceded by a comma indicate differentiation with respect to the corresponding Cartesian coordinate.

over these solutions will be *real* and nonzero if and only if $c_{ijkl}(x)$ is strongly elliptic in the sense that

$$c_{ijkl}(x)n_in_jv_kv_l > 0 \text{ whenever } v \neq 0. \tag{1.3}$$

A recent result of Toupin and Gurtin [1], which is a generalization to anisotropic materials of a theorem due to Gurtin and Sternberg [2], demonstrates the connection between wave propagation and uniqueness. This result may be stated roughly as follows: the displacement problem of linear elastodynamics has at most one solution provided c_{ijkl} is constant and strongly elliptic. Thus, if the medium is homogeneous, a sufficient condition for uniqueness is that three linearly independent acceleration waves can propagate in every direction.

The foregoing results can be easily generalized to visco-elastic solids with the aid of a correspondence principle which we prove in this paper. This principle asserts that the speed of propagation U of an acceleration wave at a point x in a viscoelastic solid with relaxation tensor $G_{ijkl}(x, t)$ and density $\rho(x)$ satisfies

$$(G_{ijkl}(x, 0)n_in_j - \rho(x)U^2\delta_{ik})a_k = 0. \tag{1.4}$$

Hence, by (1.1), this speed is identical to the speed of propagation through an elastic solid whose elasticity tensor is

$$c_{ijkl}(x) = G_{ijkl}(x, 0). \tag{1.5}$$

This correspondence principle brings out the remarkable (but intuitively obvious) fact that the speed of propagation of an acceleration wave through a viscoelastic solid depends only upon the *initial value* $G_{ijkl}(x, 0)$ of the relaxation tensor and is independent of the behavior of $G_{ijkl}(x, t)$ with t .

Assume for the remainder of this section that $G_{ijkl}(x, t)$ is *initially symmetric*, i.e.,*

$$G_{ijkl}(x, 0) = G_{klij}(x, 0). \tag{1.6}$$

Then the eigenvalue problem (1.4) has three solutions. Further these solutions will be *real* and non-zero if and only if $G_{ijkl}(x, t)$ is *initially strongly elliptic*, that is if and only if $G_{ijkl}(x, 0)$ is strongly elliptic.

Edelstein and Gurtin [3] have shown that the displacement problem of dynamic viscoelasticity theory has at most one solution provided $G_{ijkl}(x, t)$ is independent of x and initially strongly elliptic. Thus a sufficient condition for uniqueness to hold in a homogeneous viscoelastic medium is that three independent acceleration waves can propagate in every direction in the body.

The correspondence principle. The fundamental system of field equations in the linear theory of viscoelasticity consists of the following:**

$$2\epsilon_{ij}(x, t) = u_{i,j}(x, t) + u_{j,i}(x, t), \tag{2.1}$$

$$\sigma_{i,j}(x, t) + f_i(x, t) = \rho(x)u_i^{(2)}(x, t), \quad \sigma_{ij}(x, t) = \sigma_{ji}(x, t), \tag{2.2}$$

$$\sigma_{ij}(x, t) = \int_{-\infty}^t G_{ijkl}(x, t-s)\epsilon_{kl}^{(1)}(x, s) ds. \tag{2.3}$$

*Notice that $G_{ijkl}(x, 0)$ automatically has this property if it is isotropic.

**We use the notation $g^{(n)} = \partial^n g / \partial t^n$.

Here $u_i(x, t)$, $\epsilon_{ij}(x, t)$, $\sigma_{ij}(x, t)$, and $f_i(x, t)$ are the Cartesian components of the displacement vector, the strain tensor, the stress tensor, and the body force density vector defined for every pair (x, t) such that x is a point of the open region \mathcal{R} occupied by the interior of the body and t ($-\infty < t < \infty$) is the time. Further $\rho(x)$ is the mass density defined for every $x \in \mathcal{R}$, and $G_{ijkl}(x, t)$ are the components of the relaxation tensor which are defined for every $(x, t) \in \mathcal{R} \times [0, \infty)$ and satisfy

$$G_{ijkl} = G_{jikl} = G_{ijlk} . \tag{2.4}$$

The first symmetry relation follows from the symmetry of the stress tensor, while the second follows (without loss in generality) from the symmetry of the strain tensor.

We suppose once and for all that we are given a solution $u_i, \epsilon_{ij}, \sigma_{ij}$ of (2.1), (2.2), (2.3) on $\mathcal{R} \times (-\infty, \infty)$ which meets the initial condition

$$u_i(x, t) = 0, \quad (x, t) \in \mathcal{R} \times (-\infty, 0) \tag{2.5}$$

and corresponds to data ρ, G_{ijkl}, f_i which satisfies:

- (i) $\rho > 0$ is continuous;
- (ii) G_{ijkl} is twice continuously differentiable and meets (2.4);
- (iii) f_i is continuous.

Moreover we assume the existence of an *acceleration wave* propagating through \mathcal{R} —that is a one-parameter family of surfaces S_t ($0 \leq t < \infty$) which has the following properties:

- (iv) the hypersurface

$$\sum_s = \{(x, t) \mid x \in S_t, (0 \leq t < \infty)\}$$

is smooth and orientable;

- (v) given an $x \in \mathcal{R}$, the set

$$D_x = \{t \mid x \in S_t\}$$

has measure zero;

- (vi) $u_i, u_i^{(1)}, u_{i,j}$ are continuous on $\mathcal{R} \times (-\infty, \infty)$, $u_i^{(2)}, u_{i,i}^{(1)}, u_{i,jk}$ have jump discontinuities across \sum_s but are continuous everywhere else on $\mathcal{R} \times (-\infty, \infty)$, and $u_{i,jk}$ is bounded on compact subsets of $\mathcal{R} \times (-\infty, \infty)$.

Let $(x, t) \in \sum_s$. Condition (iv) then implies the existence of a normal vector ν at (x, t) . Indeed, such a vector is given by

$$\nu = (n_1, n_2, n_3, -U), \tag{2.6}$$

where n_i are the components of a unit vector which is normal to S_t at x and $U \geq 0$. The latter condition is imposed to insure the uniqueness of ν at points where $U \neq 0$. The number U is called the *speed of propagation* of S_t at x , while the unit vector n is called the *direction* of S_t at x . Condition (v) is merely the requirement that the surface S_t , as it progresses in time, does not pass through a given point x too many times. This condition together with condition (vi) implies that for any $x \in \mathcal{R}$, $u_{i,jk}(x, t)$, $u_{i,i}^{(1)}(x, t)$, and $u_i^{(2)}(x, t)$ are discontinuous only when $t \in D_x$. Finally we remark that all integrals are to be taken in the sense of Lebesgue.

We are now in a position to prove the

CORRESPONDENCE PRINCIPLE FOR WAVE PROPAGATION. *The speed of propagation U of the acceleration wave S_t at a point $x \in S_t$ is a solution of the eigenvalue problem*

$$(c_{ijkl}n_jn_l - \rho U^2 \delta_{ik})a_k = 0, \tag{2.7}$$

where

$$c_{ijkl} = G_{ijkl}(x, 0) \tag{2.8}$$

and n is the direction of S_i at x . That is, the speed of propagation is identical to the speed of propagation of S_i through an elastic solid whose elasticity tensor is c_{ijkl} .

PROOF. Integrate (2.3) by parts and use (2.1), (2.5) to verify that

$$\sigma_{ij} = c_{ijkl}\epsilon_{kl} + G_{ijkl}^{(1)} * \epsilon_{kl}, \tag{2.9}$$

where c_{ijkl} is given by (2.8) and

$$G_{ijkl}^{(1)} * \epsilon_{kl}(x, t) = \int_{-\infty}^t G_{ijkl}^{(1)}(x, t - s)\epsilon_{kl}(x, s) ds. \tag{2.10}$$

Now substitute (2.1), (2.9) into (2.2) and use the symmetries (2.4) together with (2.8) to derive the displacement equations of equilibrium in the form

$$c_{ijkl}u_{k,lj} = \rho u_i^{(2)} - F_i, \tag{2.11}$$

where

$$F_i = c_{ijkl,i}u_{k,l} + G_{ijkl,i}^{(1)} * u_{k,l} + G_{ijkl}^{(1)} * u_{k,lj} + f_i. \tag{2.12}$$

Define the functions p_{kl} and q_i through

$$p_{kl} = u_{k,l}, \quad q_i = u_i^{(1)}. \tag{2.13}$$

Then p_{kl} and q_i are continuous, have continuous derivatives except on \sum_s , and satisfy the compatibility conditions

$$p_{kl}^{(1)} = q_{k,l}. \tag{2.14}$$

Moreover, the jumps in the (four dimensional) gradients of p_{kl} and q_i are parallel to the normal ν of \sum_s and hence, because of (2.6), take the form*

$$\begin{aligned} [p_{kl,i}] &= \lambda_{kl}n_i, \\ [p_{kl}^{(1)}] &= -\lambda_{kl}U, \\ [q_{k,l}] &= a_kn_l, \\ [q_k^{(1)}] &= -a_kU. \end{aligned} \tag{2.15}$$

Thus (2.13), (2.14), (2.15), together with the continuity of c_{ijkl} and ρ across \sum_s imply

$$[c_{ijkl}u_{k,lj}] = c_{ijkl}\lambda_{kl}n_j, \quad [\rho u_i^{(2)}] = -\rho a_i U, \quad -\lambda_{kl}U = a_kn_l. \tag{2.16}$$

By (2.8) and the smoothness assumptions (ii), (iii), and (vi) it follows that

$$[c_{ijkl,i}u_{k,l}] = [G_{ijkl,i}^{(1)} * u_{k,l}] = [f_i] = 0. \tag{2.17}$$

We now prove that $[G_{ijkl}^{(1)} * u_{k,lj}] = 0$. Fix $(x, t) \in \sum_s$, let $A = G_{ijkl}^{(1)} * u_{k,lj}$, and notice that

*Here and in the sequel we use the usual notation for the jump $[g]$ of a function g across Σ_s . That is

$$[g](x, t) = \lim_{(x^+, t^+) \rightarrow (x, t)} g(x^+, t^+) - \lim_{(x^-, t^-) \rightarrow (x, t)} g(x^-, t^-),$$

where (x^+, t^+) and (x^-, t^-) lie on opposite sides of Σ_s .

$$A(x^+, t^+) = \int_0^t G_{ijkil}^{(1)}(x^+, t^+ - s)u_{k,lij}(x^+, s) ds + \int_t^{t^+} G_{ijkil}^{(1)}(x^+, t^+ - s)u_{k,lij}(x^+, s) ds. \tag{2.18}$$

Let Λ denote the compact set $\Omega \times [0, T]$, where Ω is a solid sphere centered at x and contained in \mathcal{R} while $T > t$. Then hypotheses (iv) and (vi) imply that $G_{ijkil}^{(1)}$ and $u_{k,lij}$ are bounded on Λ . Thus the second term in (2.18) tends to zero as $(x^+, t^+) \rightarrow (x, t)$. Moreover conditions (ii), (v), and (vii) yield

$$G_{ijkil}^{(1)}(x^+, t^+ - s)u_{k,lij}(x^+, s) \rightarrow G_{ijkil}^{(1)}(x, t - s)u_{k,lij}(x, s) \tag{2.19}$$

for almost every $s \leq t$, i.e., for $s \notin D_x$. Therefore the boundedness of $G_{ijkil}^{(1)}$ and $u_{k,lij}$ on Λ and Lebesgue's bounded convergence theorem imply

$$\lim_{(x^+, t^+) \rightarrow (x, t)} A(x^+, t^+) = \int_0^t G_{ijkil}^{(1)}(x, t - s)u_{k,lij}(x, s) ds. \tag{2.20}$$

Clearly the same result follows for $\lim_{(x^-, t^-) \rightarrow (x, t)} A(x^-, t^-)$ and thus

$$[G_{ijkil}^{(1)} * u_{k,lij}] = 0. \tag{2.21}$$

Equations (2.12), (2.17), and (2.22) imply $[F_i] = 0$. It therefore follows from (2.11), (2.16) that

$$(c_{ijkil}n_jn_l - \rho U^2 \delta_{ik})a_k = 0, \tag{2.22}$$

and this completes the proof.

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IRREVERSIBLE THERMODYNAMICS OF PERFECT PLASTICITY*

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1. Introduction. For the solution of boundary value problems in continuum mechanics, it is necessary to supplement the equation of continuity that expresses the conservation of mass and the equation of motion with the constitutive equation for the material under consideration. In the case of plasticity, the constitutive equation introduces the mechanical properties of the material by means of the relation between the

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