THE SEMI-INFINITE ELASTIC STRIP*

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1. Introduction. We consider the two-dimensional problem of a semi-infinite elastic strip occupying the region $|y| \leq 1$, $0 \leq x \leq \infty$, where x and y are rectangular Cartesian coordinates. Let σ_x, σ_y and τ be components of stress and u and v be components of displacement. We take the edges $y = \pm 1$ to be free of stress

$$y = \pm 1, \quad 0 < x < \infty: \quad \sigma_y = \tau = 0. \tag{1.1}$$

At the edge $x = 0$, one of the following conditions holds:

$$\sigma_x = \sigma_{xb}(y), \quad v = v_b(y), \tag{1.2a}$$

$$\tau = \tau_b(y), \quad u = u_b(y), \tag{1.2b}$$

$$\sigma_x = \sigma_{xb}(y), \quad \tau = \tau_b(y), \tag{1.2c}$$

$$u = u_b(y), \quad v = v_b(y) \tag{1.2d}$$

where the subscript b indicates a given function. We also prescribe that all stress and displacement components approach zero as x becomes large:

$$\text{solution} \to 0 \text{ as } x \to \infty. \tag{1.3}$$

The equations of equilibrium and compatibility are

$$\frac{\partial \tau}{\partial y} + \frac{\partial \sigma_x}{\partial x} = 0, \quad \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau}{\partial x} = 0, \tag{1.4}$$

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(\sigma_x + \sigma_y) = 0. \tag{1.5}$$

We will refer to problems (1.2a) and (1.2b) as the mixed problems, (1.2c) as the stress problem and (1.2d) as the displacement problem. These problems are solved in what follows by making use of an expansion in terms of four-component eigenvectors, each of which satisfies Eqs. (1.1), (1.3), (1.4) and (1.5). The satisfaction of boundary conditions (1.2) is accomplished by identifying the coefficients of the expansion with the coefficients of the expansion of the given boundary functions. The procedure here is formal. Readers are referred to [1] for a discussion of convergence of the expansion.

In addition to being of interest to plane elasticity theory, this problem has been shown by recent work on the foundations of the theory of thin plates and shells [2, 3, 4] to be fundamental in the analysis of the boundary layer contribution to the solution of these problems.

2. Formulation of boundary value problem.   The usual procedure is to derive from the system (1.4) and (1.5) a single fourth order differential equation for Airy's stress function. We will find it convenient instead to work with an equivalent system of four first order equations. This idea is, in fact, the heart of the method of solution presented here.

We introduce a new function $p(x, y)$ satisfying

$$\frac{\partial \sigma_x}{\partial y} - \frac{\partial p}{\partial x} = 0. \tag{2.1}$$

Using Eqs. (1.4) and (2.1), we may write the compatibility equation (1.5) in the following form:

$$\frac{\partial^2}{\partial x^2} (2\sigma_x + \sigma_y) + \frac{\partial^2 p}{\partial x \partial y} = 0. \tag{2.2}$$

Equation (2.2) can be integrated with respect to x to yield

$$\frac{\partial p}{\partial y} + \frac{\partial}{\partial x} (2\sigma_x + \sigma_y) = 0, \tag{2.3}$$

where the function of integration is set to zero to satisfy condition (1.3).

The introduction of the variable p and the derivation of the compatibility equation in the form (2.3) allow us to write the system (1.4), (2.1) and (2.3) in the vector form

$$\frac{\partial \mathbf{f}}{\partial y} + U \cdot \frac{\partial \mathbf{f}}{\partial x} = 0, \tag{2.4}$$

where

$$\mathbf{f} = \begin{bmatrix} \sigma_y \\ -\tau \\ \sigma_x \\ p \end{bmatrix}, \qquad U = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 2 & 0 \end{bmatrix}. \tag{2.5}, (2.6)$$

The form (2.4) of the basic equation is motivated by the fact that the ordinary differential equation obtained upon using the separation of variables procedure will be in a standard vector form.

We next examine the boundary conditions on vector $\mathbf{f}$. Two of the stress-strain relations are:

$$\frac{\partial u}{\partial x} = \frac{1}{E} (\sigma_x - \nu \sigma_y), \tag{2.7}$$

$$\frac{\partial v}{\partial y} = \frac{1}{E} (\sigma_y - \nu \sigma_x), \tag{2.8}$$

where ν is Poisson's ratio and E is Young's modulus. From Eqs. (1.4), (2.1) and (2.7) we obtain

$$\frac{\partial^2 u}{\partial x \partial y} = \frac{1}{E} \left(\frac{\partial p}{\partial x} + \nu \frac{\partial \tau}{\partial x} \right). \tag{2.9}$$

Integration of Eq. (2.9) and use of Eq. (1.3) yields

$$\frac{\partial u}{\partial y} = \frac{1}{E}\left(p + \nu\tau\right). \tag{2.10}$$

Using Eqs. (2.8) and (2.10) we see that the boundary conditions (1.2) are equivalent, respectively, to specification on edge $x = 0$ of

$$f^{(1)} = \sigma_{yb} = \nu\sigma_{xb} + E\frac{dv_b}{dy}, \qquad f^{(3)} = \sigma_{xb}, \tag{2.11a}$$

$$f^{(2)} = -\tau_b, \qquad f^{(4)} = p_b = -\nu\tau_b + E\frac{du_b}{dy}, \tag{2.11b}$$

$$f^{(2)} = -\tau_b, \qquad f^{(3)} = \sigma_{xb}, \tag{2.11c}$$

$$f^{(1)} - \nu f^{(3)} = \sigma_{yb} - \nu\sigma_{xb} = E\frac{dv_b}{dy},$$

$$f^{(4)} - \nu f^{(2)} = p_b + \nu\tau_b = E\frac{du_b}{dy}, \tag{2.11d}$$

Conditions (1.1) and (1.3) take the form

$$x \to \infty, \qquad |y| < 1: \qquad \mathbf{f} \to 0 \tag{2.12}$$

$$0 < x < \infty, \qquad y = \pm 1: \qquad f^{(1)} = f^{(2)} = 0 \tag{2.13}$$

We seek a representation of the solution $\mathbf{f}$ in the interior of the strip in terms of its value $\mathbf{f}_b$ on the end. We will then show how to proceed from this representation to the solution of problems with end conditions (1.2).

3. Series solution. We take the solution in the form of an infinite series

$$\mathbf{f} = \sum_n c_n \mathbf{Z}_n(y)e^{i\alpha_n x}, \tag{3.1}$$

where c_n and α_n are constants to be determined as are the vector functions $\mathbf{Z}_n$. Substitution of the series (3.1) into the differential Eq. (2.4) yields the following system of ordinary differential equations for the functions $\mathbf{Z}_n$:

$$\mathbf{Z}_n' + i\alpha_n U \cdot \mathbf{Z}_n = 0, \tag{3.2}$$

where the prime denotes differentiation with respect to y. From conditions (2.13) we obtain the following boundary conditions on the functions $\mathbf{Z}_n$:

$$y = \pm 1: \qquad Z_n^{(1)} = Z_n^{(2)} = 0 \tag{3.3}$$

System (3.2) with the boundary conditions (3.3) forms an eigenvalue problem. Its solution is elementary as the system (3.2) has constant coefficients. The eigenvectors divide themselves into two sets, those where the first component $Z_n^{(1)}$ is odd in y, and those where it is even.

The odd eigenvectors are given by

$$\begin{aligned}
{}_0Z_n^{(1)} &= -{}_0\alpha_n^2[{}_0\alpha_n y \sinh {}_0\alpha_n \cosh {}_0\alpha_n y - {}_0\alpha_n \cosh {}_0\alpha_n \sinh {}_0\alpha_n y], \\
{}_0Z_n^{(2)} &= i{}_0\alpha_n^2[{}_0\alpha_n y \sinh {}_0\alpha_n \sinh {}_0\alpha_n y - ({}_0\alpha_n \cosh {}_0\alpha_n - \sinh {}_0\alpha_n) \cosh {}_0\alpha_n y], \\
{}_0Z_n^{(3)} &= {}_0\alpha_n^2[{}_0\alpha_n y \sinh {}_0\alpha_n \cosh {}_0\alpha_n y - ({}_0\alpha_n \cosh {}_0\alpha_n - 2\sinh {}_0\alpha_n) \sinh {}_0\alpha_n y], \\
{}_0Z_n^{(4)} &= -i{}_0\alpha_n^2[{}_0\alpha_n y \sinh {}_0\alpha_n \sinh {}_0\alpha_n y - ({}_0\alpha_n \cosh {}_0\alpha_n - 3\sinh {}_0\alpha_n) \cosh {}_0\alpha_n y],
\end{aligned} \tag{3.4}$$

while the even eigenvectors are given by

$$_eZ_n^{(1)} = -_e\alpha_n^2[_e\alpha_n y \cosh {}_e\alpha_n \sinh {}_e\alpha_n y - {}_e\alpha_n \sinh {}_e\alpha_n \cosh {}_e\alpha_n y],$$

$$_eZ_n^{(2)} = i_e\alpha_n^2[_e\alpha_n y \cosh {}_e\alpha_n \cosh {}_e\alpha_n y - (_e\alpha_n \sinh {}_e\alpha_n - \cosh {}_e\alpha_n) \sinh {}_e\alpha_n y],$$

$$_eZ_n^{(3)} = {}_e\alpha_n^2[_e\alpha_n y \cosh {}_e\alpha_n \sinh {}_e\alpha_n y - (_e\alpha_n \sinh {}_e\alpha_n - 2 \cosh {}_e\alpha_n) \cosh {}_e\alpha_n y],$$

$$_eZ_n^{(4)} = -i_e\alpha_n^2[_e\alpha_n y \cosh {}_e\alpha_n \cosh {}_e\alpha_n y - (_e\alpha_n \sinh {}_e\alpha_n - 3 \cosh {}_e\alpha_n) \sinh {}_e\alpha_n y].$$

(3.5)

The eigenvalues $_0\alpha_n$ and $_e\alpha_n$ are solutions of the following transcendental equations:

$$\sinh (2_0\alpha_n) = 2_0\alpha_n, \qquad \sinh (2_e\alpha_n) = -2_e\alpha_n.$$

(3.6)

Equations (3.6) have been investigated[1] by Hillman and Salzer [10] and by Robbins and Smith [11]. These equations have infinite roots in each quadrant of the complex α-plane. To make the solution (3.1) satisfy condition (2.12), we use only eigenfunctions in (3.1) associated with roots α_n in the upper half-plane. Hence, the sum (3.1) is extended over all eigenvectors (3.4) and (3.5) whose eigenvalues have positive imaginary parts. We note that these occur in pairs in the first and second quadrants such that, if α_n is a root, so[2] is $-\alpha_n^*$.

It remains to discuss the satisfaction of the boundary conditions along the edge $x = 0$. Letting x be zero in Eq. (3.1) yields

$$\sum_n c_n Z_n(y) = \mathbf{f}_b,$$

(3.7)

where $\mathbf{f}_b$ is the vector of end values of $\mathbf{f}$. From Eq. (3.7) we can determine the constants c_n in terms of end values $\mathbf{f}_b$. Note that the vectors Z_n do not satisfy the condition of orthogonality,[3]

$$\int_{-1}^{+1} Z_k^\dagger \cdot U \cdot Z_n \, dy = 0, \qquad k \neq n.$$

(3.8)

However, there exists a set of vectors $\mathbf{W}_n$ which are biorthogonal to Z_n and satisfy the condition

$$\int_{-1}^{+1} \mathbf{W}_k^\dagger \cdot U \cdot Z_n \, dy = 0, \qquad k \neq n,$$

(3.9)

and these are obtained in Section 4. Proceeding formally, we premultiply both sides of Eq. (3.7) by $\mathbf{W}_n^\dagger \cdot U$, integrate and use Eq. (3.9) to obtain

$$c_n = K_n^{-1} \int_{-1}^{+1} \mathbf{W}_n^\dagger \cdot U \cdot \mathbf{f}_b \, dy, \qquad K_n = \int_{-1}^{+1} \mathbf{W}_n^\dagger \cdot U \cdot Z_n \, dy$$

(3.10)

In expanded form, Eq. (3.10) gives

$$c_n = K_n^{-1} \int_{-1}^{+1} [W_n^{*(4)}f_b^{(1)} - W_n^{*(1)}f_b^{(2)} + (2W_n^{*(4)} - W_n^{*(2)})f_b^{(3)} - W_n^{*(3)}f_b^{(4)}] \, dy,$$

(3.11)

$$K_n = \int_{-1}^{+1} [W_n^{*(4)}Z_n^{(1)} - W_n^{*(1)}Z_n^{(2)} + (2W_n^{*(4)} - W_n^{*(2)})Z_n^{(3)} - W_n^{*(3)}Z_n^{(4)}] \, dy.$$

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[1]To obtain the equations considered by these writers we substitute $s_n = i\alpha_n^*$, where the ( )* denotes the complex conjugate. Equations (3.6) become $\sin (2_0s_n) = 2_0s_n$ and $\sin (2_es_n) = -2_es_n$ respectively.

[2]( )* denotes the complex conjugate.

[3]A superscript T indicates the transpose of a vector or matrix. ( )† denotes the conjugate transpose.

With c_n given by Eqs. (3.11), Eq. (3.1) gives the desired representation of $\mathbf{f}$ in terms of its boundary values $\mathbf{f}_b$. In section 5 we show how to proceed from this representation to the satisfactoin of boundary conditions of the form (1.2). Note that the differential equations and the boundary conditions (1.1) and (1.3) are satisfied term by term by the series (3.1).

4. The biorthogonal vectors $\mathbf{W}_m$. The vectors $\mathbf{W}_m$ satisfy the adjoint equation

$$\mathbf{W}_n' + i\alpha_n^* U^\dagger \cdot \mathbf{W}_n = 0. \tag{4.1}$$

Boundary conditions on $\mathbf{W}_n$ are to be chosen so that the following equation is satisfied:

$$\int_{-1}^{+1} (\mathbf{W}_k^\dagger \cdot \mathbf{Z}_i' + \mathbf{W}_k^{\dagger'} \cdot \mathbf{Z}_i)\, dy = \int_{-1}^{+1} (\mathbf{W}_k^\dagger \cdot \mathbf{Z}_i)'\, dy = [\mathbf{W}_k^\dagger \cdot \mathbf{Z}_i]_{-1}^{+1} = 0. \tag{4.2}$$

Since $\mathbf{Z}_n$ satisfies the conditions (3.3), we use the following boundary conditions for $\mathbf{W}_n$ in order that Eq. (4.2) will hold:

$$y = \pm 1: \qquad W_n^{(3)} = W_n^{(4)} = 0. \tag{4.3}$$

Substituting into Eq. (4.2) from Eqs. (3.2) and (4.1), we obtain

$$\int_{-1}^{+1} [\mathbf{W}_k^\dagger \cdot (-i\alpha_i U \cdot Z_i) + (i\alpha_k(U^\dagger \cdot \mathbf{W}_k)^\dagger) \cdot \mathbf{Z}_i]\, dy = 0, \tag{4.4}$$

or

$$(\alpha_k - \alpha_i) \int_{-1}^{+1} \mathbf{W}_k^\dagger \cdot U \cdot \mathbf{Z}_i\, dy = 0, \tag{4.5}$$

which yields the biorthogonality condition (3.9).

The eigenvectors $\mathbf{W}_n$ are obtained by solving the eigenvalue problem given by Eqs. (4.1) and (4.3). The eigenvectors again split into two sets. Corresponding to the eigenvalues given by Eq. (3.6a) we obtain:

$$
\begin{aligned}
{}_0W_n^{(1)} &= -i[{}_0\alpha_n^* y \sinh {}_0\alpha_n^* \sinh {}_0\alpha_n^* y - ({}_0\alpha_n^* \cosh {}_0\alpha_n^* + \sinh {}_0\alpha_n^*) \cosh {}_0\alpha_n^* y], \\
{}_0W_n^{(2)} &= [{}_0\alpha_n^* y \sinh {}_0\alpha_n^* \cosh {}_0\alpha_n^* y - ({}_0\alpha_n^* \cosh {}_0\alpha_n^* + 2\sinh {}_0\alpha_n^*) \sinh {}_0\alpha_n^* y], \\
{}_0W_n^{(3)} &= -i[{}_0\alpha_n^* y \sinh {}_0\alpha_n^* \sinh {}_0\alpha_n^* y - ({}_0\alpha_n^* \cosh {}_0\alpha_n^* - \sinh {}_0\alpha_n^*) \cosh {}_0\alpha_n^* y], \\
{}_0W_n^{(4)} &= [{}_0\alpha_n^* y \sinh {}_0\alpha_n^* \cosh {}_0\alpha_n^* y - {}_0\alpha_n^* \cosh {}_0\alpha_n^* \sinh {}_0\alpha_n^* y].
\end{aligned} \tag{4.6}
$$

The eigenvectors which correspond to the eigenvalues given by Eq. (3.6b) are

$$
\begin{aligned}
{}_eW_n^{(1)} &= -i[{}_e\alpha_n^* y \cosh {}_e\alpha_n^* \cosh {}_e\alpha_n^* y - ({}_e\alpha_n^* \sinh {}_e\alpha_n^* + \cosh {}_e\alpha_n^*) \sinh {}_e\alpha_n^* y], \\
{}_eW_n^{(2)} &= [{}_e\alpha_n^* y \cosh {}_e\alpha_n^* \sinh {}_e\alpha_n^* y - ({}_e\alpha_n^* \sinh {}_e\alpha_n^* + 2\cosh {}_e\alpha_n^*) \cosh {}_e\alpha_n^* y], \\
{}_eW_n^{(3)} &= -i[{}_e\alpha_n^* y \cosh {}_e\alpha_n^* \cosh {}_e\alpha_n^* y - ({}_e\alpha_n^* \sinh {}_e\alpha_n^* - \cosh {}_e\alpha_n^*) \sinh {}_e\alpha_n^* y], \\
{}_eW_n^{(4)} &= [{}_e\alpha_n^* y \cosh {}_e\alpha_n^* \sinh {}_e\alpha_n^* y - {}_e\alpha_n^* \sinh {}_e\alpha_n^* \cosh {}_e\alpha_n^* y].
\end{aligned} \tag{4.7}
$$

Substitution from Eqs. (3.4) and (4.6) into Eq. (3.11) and integration yields

$$
{}_0K_n = 8({}_0\alpha_n)^2 \sinh^4 {}_0\alpha_n. \tag{4.8}
$$

Similarly, Eqs. (3.5) and (4.7) yield

$$
{}_eK_n = 8({}_e\alpha_n)^2 \cosh^4 {}_e\alpha_n. \tag{4.9}
$$

5. Boundary conditions on edge $x = 0$. We wish to show how the constants c_n are determined in terms of the given functions of Eqs. (2.11). For the mixed problems, (2.11a) and (2.11b), this determination can be done in an explicit manner. For problem (2.11a), we set

$$f_b^{(1)} = \sigma_{yb}, \qquad f_b^{(2)} = \sum c_n Z_n^{(2)}(y), \qquad f_b^{(3)} = \sigma_{xb}, \qquad f_b^{(4)} = \sum c_n Z_n^{(4)}(y). \qquad (5.1)$$

Substitution of Eqs. (5.1) into Eq. (3.11) for c_n yields

$$c_n = K_n^{-1} \int_{-1}^{+1} [W_n^{*(4)} \sigma_{yb} + (2W_n^{*(4)} - W_n^{*(2)})\sigma_{xb}] \, dy$$

$$- K_n^{-1} \sum_i c_i \int_{-1}^{+1} [W_n^{*(1)} Z_i^{(2)} + W_n^{*(3)} Z_i^{(4)}] \, dy. \qquad (5.2)$$

The functions $Z_n^{(2)}$ and $Z_n^{(4)}$ form the components of a two-dimensional vector whose biorthogonal vector has components $W_n^{(1)}$ and $W_n^{(3)}$. Substitution from Eqs. (3.4), (3.5), (4.6) and (4.7) shows that

$$\int_{-1}^{+1} [W_n^{*(1)} Z_i^{(2)} + W_n^{*(3)} Z_i^{(4)}] \, dy = \begin{cases} 0, & n \neq j, \\ -\tfrac{1}{2} K_n, & n = j. \end{cases} \qquad (5.3)$$

Utilizing Eq. (5.3), we can rewrite Eq. (5.2) in the form

$$c_n = 2K_n^{-1} \int_{-1}^{+1} [W_n^{*(4)} \sigma_{yb} + (2W_n^{*(4)} - W_n^{*(2)})\sigma_{xb}] \, dy, \qquad (5.4)$$

which gives the constants c_n directly from the given boundary functions σ_{yb} and σ_{xb} .

The other mixed problem (2.11b) is solved in a similar manner and also involves an inner biorthogonality. The result is

$$c_n = 2K_n^{-1} \int_{-1}^{+1} [W_n^{*(1)} \tau_b - W_n^{*(3)} p_b] \, dy. \qquad (5.5)$$

The stress and displacement problems do not exhibit an inner biorthogonality property and, therefore, do not reduce to a form from which the coefficients c_n may be directly obtained. For problem (2.11c) we obtain

$$\begin{aligned} f_b^{(1)} &= \sum c_n Z_n^{(1)}(y), & f_b^{(2)} &= -\tau_b , \\ f_b^{(3)} &= \sigma_{xb} , & f_b^{(4)} &= \sum c_n Z_n^{(4)}(y). \end{aligned} \qquad (5.6)$$

Substitution into Eq. (3.11) yields

$$c_n = K_n^{-1} \sum_i c_i \int_{-1}^{+1} [W_n^{*(4)} Z_i^{(1)} - W_n^{*(3)} Z_i^{(4)}] \, dy + F_n , \qquad (5.7)$$

where

$$F_n = K_n^{-1} \int_{-1}^{+1} [W_n^{*(1)} \tau_b + (2W_n^{*(4)} - W_n^{*(2)})\sigma_{xb}] \, dy. \qquad (5.8)$$

Making use of Eqs. (3.4), (3.5) and (4.6) through (4.9), we may carry out the integration in (5.7) to obtain

$$c_n = \sum_i R_{ni} c_i + 2F_n , \qquad n = 1, 2, 3, \cdots , \qquad (5.9)$$

where

$$R_{ni} = \begin{cases} \dfrac{-4\,_0\alpha_j^3(_0\alpha_n^2 \sinh^2 {}_0\alpha_j - {}_0\alpha_j^2 \sinh^2 {}_0\alpha_n)}{(_0\alpha_j + {}_0\alpha_n)(_0\alpha_j^2 - {}_0\alpha_n^2)^2 \sinh^4 {}_0\alpha_n}, & n \neq j, \quad n \text{ and } j \text{ both odd}, \\[2ex] \dfrac{4\,_e\alpha_j^3(_e\alpha_n^2 \cosh^2 {}_e\alpha_j - {}_e\alpha_j^2 \cosh^2 {}_e\alpha_n)}{(_e\alpha_j + {}_e\alpha_n)(_e\alpha_j^2 - {}_e\alpha_n^2)^2 \cosh^4 {}_e\alpha_n}, & n \neq j, \quad n \text{ and } j \text{ both even}, \\[2ex] 0, & n = j \text{ or } n \neq j \text{ with } n \text{ odd, } j \text{ even or } n \text{ even, } j \text{ odd}. \end{cases} \qquad (5.10)$$

Equation (5.9) is a system of infinite equations in infinitely many unknowns, which can be solved by truncation to obtain values of c_n to any desired degree of accuracy.

For the displacement problem, (2.11d), we write

$$f_b^{(1)} = E v_b' + \nu \sum_n c_n Z_n^{(3)}, \qquad f_b^{(2)} = \sum_n c_n Z_n^{(2)},$$

$$f_b^{(3)} = \sum_n c_n Z_n^{(3)}, \qquad\qquad f_b^{(4)} = E u_b' + \nu \sum_n c_n Z_n^{(2)},$$

which leads to a system of equations for the c_i's of the same form as (5.9).

6. Convergence of series expansions. For the series expansions to converge to the given boundary functions, it is necessary that the boundary functions be self-equilibrating:

$$\int_{-1}^{+1} f_b^{(2)}\, ds = \int_{-1}^{+1} f_b^{(3)}\, ds = \int_{-1}^{+1} f_b^{(3)} s\, ds = \int_{-1}^{+1} (s^2 - 1) f_b^{(4)}\, ds = 0. \qquad (6.1)$$

The meaning of restrictions (6.1) is made clear by showing that they must be satisfied by any solution of Eqs. (1.1), (1.3), (1.4) and (1.5) for $0 < x < \infty$ and as $x \to 0$. Static equilibrium considerations[1] show that the net force and moment on any section vanish for all x,

$$\int_{-1}^{+1} \sigma_x\, dy = \int_{-1}^{+1} \sigma_x y\, dy = \int_{-1}^{+1} \tau\, dy = 0, \qquad 0 < x < \infty, \qquad (6.2)$$

which are just the first three of conditions (6.1). Integration by parts gives

$$\frac{\partial}{\partial x} \int_{-1}^{+1} (y^2 - 1)p\, dy = \int_{-1}^{+1} (y^2 - 1) \frac{\partial \sigma_x}{\partial y}\, dy = -2 \int_{-1}^{+1} y\sigma_x\, dy \qquad (6.3)$$

which vanishes by (6.2). Hence,

$$\int_{-1}^{+1} (y^2 - 1)p\, dy = \text{constant} = 0 \qquad (6.4)$$

for all $x > 0$ if we use (1.3). Condition (6.4) agrees with condition (6.1)$_4$.

Conditions in addition to (6.1) that are sufficient to ensure uniform convergence of the series to the given boundary values are discussed in [1]. We here examine numerically the convergence for a particular example of the stress problem. Consider the boundary stresses

$$\tau_b = -\sin \pi y, \qquad \sigma_{xb} = -3y + 10y^3 - 7y^5, \qquad (6.5)$$

which satisfy the first three conditions of (6.1). A CDC 1604 computer was programmed to solve for the vector F_n from (5.8) and the matrix R_{ni} from (5.10). The linear system

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[1] This result can also be obtained from consideration of the basic equations (1.1), (1.3), (1.4) and (1.5).

TABLE I

First quadrant eigenvalues from [10] and [11].

n	Re $_0\alpha_n$	Im $_0\alpha_n$	Re $_e\alpha_n$	Im $_e\alpha_n$
1	1.384339	3.748838	1.125365	2.106196
2	1.676105	6.949980	1.551575	5.356269
3	1.858384	10.119259	1.775544	8.536683
4	1.991571	13.277274	1.929405	11.699178
5	2.096626	16.429872	2.046853	14.854060
6	2.183398	19.579409	2.141891	18.004933
7	2.257320	22.727036	2.221723	21.153414
8	2.321714	25.873384	2.290553	24.300342
9	2.378758	29.018831	2.351048	27.446203
10	2.429959	32.163617	2.405013	30.591295

Asymptotic formulae are as follows: Re $_0\alpha_n \sim .5 \log_e (4n + 1)\pi$,
Im $_n\alpha_0 \sim (n + .25)\pi$, Re $_e\alpha_n \sim .5 \log_e (4n + 3)\pi$,
Im $_e\alpha_n \sim (n + .75)\pi$

(5.9) was then solved and the coefficients c_n were used to check the values of τ and σ_s on the boundary $x = 0$ given by the series against the specified functions (6.5). The eigenvalues are given in Table 1. The results for five and ten sets of paired eigenvalues (in the first and second quadrants) are shown in Figures 1 and 2 and show a rapid rate of convergence. Note from (5.10) that Eqs. (5.9) uncouple into two sets, one set associated with odd eigenvalues and one set with even. The authors found the calculations simple to make because the program could be written in complex arithmetic.
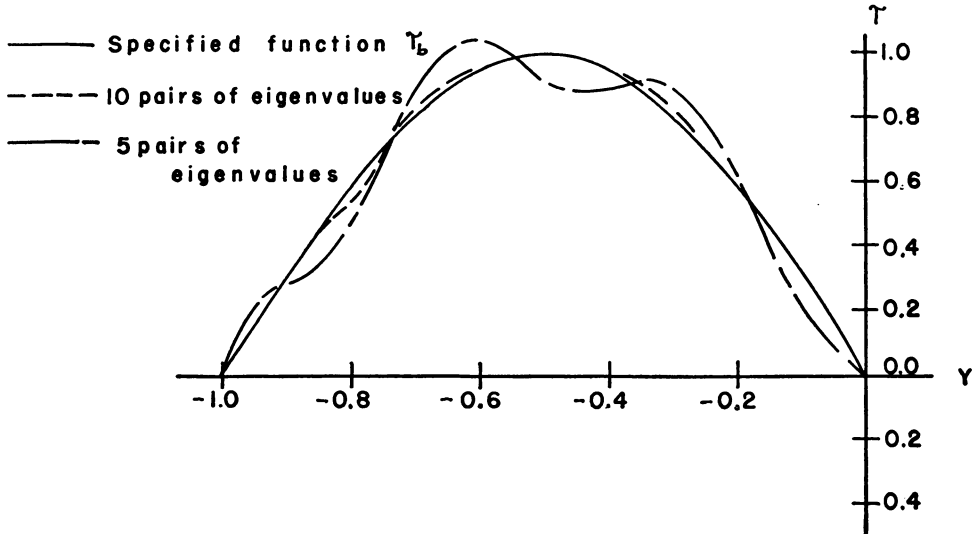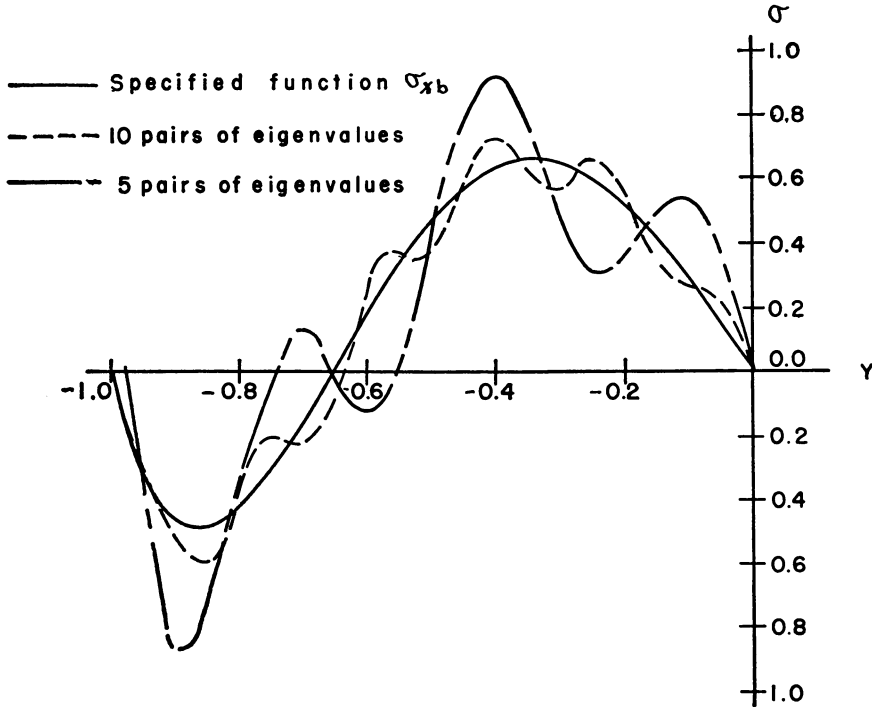

FIG. 1.

Fɪɢ. 2.

7. Concluding remarks. Smith [5] considered the problem of the bending of a semi-infinite strip loaded by moments along the short edge, his problem being mathematically equivalent to our mixed problems. Smith obtains a solution of his problem directly in a two-component vector form. Eqs. (5.4) and (5.5) agree with the corresponding results of Smith if the eigenvalues are carefully compared[1]. We note that Smith's form of the solution can also be obtained, for the mixed problems, by use of the Fourier integral [8]. The mixed problems have also been considered by Horvay [6] and by Horvay and Born [7] by another method. Their method is extended to the stress problem, but we believe that the method given here provides a more simple[2] and accurate numerical method. At the same time, it answers the question of why the stress and displacement problems are so much harder to solve than the mixed problems.

Note added in revision. Since this paper was first submitted a paper has appeared [12] which treats the stress and displacement problems by means of the Laplace transform and the calculus of residues. The resulting series expansions are similar in form to those obtained here, but are not identical since the unknown boundary functions are expanded by means of trigonometric series rather than in terms of eigenfunctions Z_n. Our approach shows that the series expansions can be directly obtained by a generalized Fourier analysis without appeal to transform methods.

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[1]See footnote after Eq. (3.6).
[2]See footnote on Page 71 of [6].

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