EXTENDED THEOREMS OF LIMIT ANALYSIS*

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1. Introduction. In limit analysis, the statically admissible stress field cannot lie outside of the hypersurface of yield criterion, and the stress field calculated by the kinematically admissible velocity field should be on the hypersurface [1]. In this paper such a requirement is eliminated, and replaced by the integral mean of the yield criterion.

The material considered is perfectly plastic and isotropic. As far as the proofs of these generalized problems are concerned, the discussion is restricted to continuous stress and velocity fields for the sake of brevity.

Mura and Lee [2] showed that a state of impeding plastic flow renders the following functional stationary, the safety factor being the stationary value of the functional. The functional is:

$$F[v_i , s_{ij} , \sigma, R_i , m, \mu, \varphi] = \int_V s_{ij}\tfrac{1}{2}(v_{i,j} + v_{j,i})\, dV$$

$$+ \int_V \sigma\delta_{ij}v_{i,j}\, dV - \int_{S_V} R_iv_i\, dS - m\left(\int_{S_T} T_iv_i\, dS - 1\right) - \int_V \mu[f(s_{ij}) + \varphi^2]\, dV \tag{1}$$

with constraint condition

$$\mu \geq 0. \tag{2}$$

The arguments of F are the independent variables: velocity v_i , stress deviation s_{ij} , and Lagrangian multipliers σ, $R_i m$, μ and φ. The function $f(s_{ij})$ is the yield criterion

$$f(s_{ij}) = \tfrac{1}{2}s_{ij}s_{ij} - k^2 , \tag{3}$$

and T_i is the given surface traction defined on a part of the boundary surface denoted by S_T . On S_V , the remaining part of the boundary, the velocity vector is required to vanish.

Setting the variation of (1) equal to zero yields the following conditions:

$$\begin{aligned} \tfrac{1}{2}(v_{i,j} + v_{j,i}) &= \mu \frac{\partial f}{\partial s_{ij}} \quad \text{in} \quad V, \\ \mu &\geq 0, \end{aligned} \tag{4}$$

$$(s_{ij} + \delta_{ij}\sigma)_{,j} = 0 \qquad \text{in} \quad V, \tag{5}$$

$$(s_{ij} + \delta_{ij}\sigma)n_j = mT_i \qquad \text{on} \quad S_T , \tag{6}$$

$$(s_{ij} + \delta_{ij}\sigma)n_j = R_i \qquad \text{on} \quad S_V , \tag{7}$$

$$f(s_{ij}) + \varphi^2 = 0 \qquad \text{in} \quad V, \tag{8}$$

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$$\mu\varphi = 0 \qquad \text{in} \quad V, \tag{9}$$

$$\delta_{ij}v_{i,j} = \qquad \text{in} \quad V, \tag{10}$$

$$v_i = 0 \qquad \text{on} \quad S_V\ , \tag{11}$$

$$\int_{S_T} T_i v_i \, dS = 1. \tag{12}$$

It should be noted that the Lagrangian multipliers σ, R_i, m, μ and φ are respectively the mean stress, the reaction on S_V, the safety factor, the positive scalar of proportionality and the yield parameter. When the parameter φ is non-zero, $\mu = 0$ and $f(s_{ij}) < 0$; when φ vanishes, $f(s_{ij}) = 0$. It is of interest to note that (12) satisfies the requirement of classical upper bound theorem that the integral be positive definite. Setting the integral equal to unity only determines the scale of the otherwise arbitrary size of the velocity vector.

Consider the arbitrary arguments

$$v_i^0 = v_i + \delta v_i\ ,\ s_{ij}^0 = s_{ij} + \delta s_{ij}\ ,\ \cdots\ , \tag{13}$$

in which v_i, s_{ij}, $\cdots$ denote the stationary set of arguments of (1) and δv_i, δs_{ij}, etc. are the variations. If the arguments of (1) are substituted by (13), giving regards to conditions (4) to (12) for v_i, s_{ij}, etc., F takes the form

$$\begin{aligned} F[v_i^0\ , s_{ij}^0\ , \sigma^0\ , R_i^0\ , m^0\ , \mu^0\ , \varphi^0] &= m + \int_V \delta s_{ij}\tfrac{1}{2}(\delta v_{i,j} + \delta v_{j,i})\, dV \\ &+ \int_V \delta\sigma\,\delta_{ij}\,\delta v_{i,j}\, dV - \int_{S_V} \delta R_i\,\delta v_i\, dS - \delta m \int_{S_T} T_i\,\delta v_i\, dS \\ &- \int_V \mu\{\tfrac{1}{2}\delta s_{ij}\delta s_{ij} + (\delta\varphi)^2\}\, dV - \int_V \delta\mu\{f(s_{ij}^0) + (\varphi^0)^2\}\, dV. \end{aligned} \tag{14}$$

2. Lower bound theorem. The inequality

$$\frac{m^0}{1 + \max\{f(s_{ij}^0) + (\varphi^0)^2\}/2k^2} \le m \tag{15}$$

holds for any set of s_{ij}^0, σ^0, m^0, μ^0 and φ^0 satisfying

$$(s_{ij}^0 + \delta_{ij}\sigma^0)_{,j} = 0 \qquad \text{in} \quad V, \tag{16}$$

$$(s_{ij}^0 + \delta_{ij}\sigma^0)n_j = m^0 T_i \quad \text{on} \quad S_T\ , \tag{17}$$

$$\int_V \mu^0\{f(s_{ij}^0) + (\varphi^0)^2\}\, dV = 0, \tag{18}$$

$$\mu^0 \ge 0. \tag{19}$$

Since the right side of (15) is the safety factor, the left side gives a lower bound for the safety factor.

The theorem is proven as follows. Expression (14) can be transformed into

$$F = m - \int_V \mu\{\tfrac{1}{2}\delta s_{ij}\delta s_{ij} + (\delta\varphi)^2\}\, dV - \int_V \delta\mu\{f(s_{ij}^0) + (\varphi^0)^2\}\, dV \tag{20}$$

by integrating the second and third terms by parts in view of (5), (6), (7), (16) and (17) and setting

$$(s_{ij}^0 + \delta_{ij}\sigma^0)n_j = R_i^0 \quad \text{on} \quad S_V , \tag{21}$$

where R_i^0 denotes the reaction of the stress field on S_V. Also, integrating (1) with arbitrary arguments v_i^0, s_{ij}^0, σ^0, R_i^0, m^0, μ^0 and φ^0 and constraint conditions (16), (17) and (21) gives

$$F = m^0 - \int_V \mu^0\{f(s_{ij}^0) + (\varphi^0)^2\}\, dV. \tag{22}$$

Equations (18), (20) and (22) yield

$$m^0 \leq m - \int_V \delta\mu\{f(s_{ij}^0) + (\varphi^0)^2\}\, dV, \tag{23}$$

because of (18) and since $\int_V \mu\{\frac{1}{2}\ \delta s_{ij}\ \delta s_{ij} + (\delta\varphi)^2\}\ dV$ is positive definite. Condition (18) also gives

$$-\int_V \delta\mu\{f(s_{ij}^0) + (\varphi^0)^2\}\, dV = \int_V \mu\{f(s_{ij}^0) + (\varphi^0)^2\}\, dV, \tag{24}$$

since $\mu^0 = \mu + \delta\mu$. Substituting (24) into (23) and taking the maximum value of the integrand we have

$$m^0 \leq m + \max\,\{f(s_{ij}^0) + (\varphi^0)^2\} \int_V \mu\, dV. \tag{25}$$

Since

$$m = m \int_{S_T} T_i v_i\, dS = \int_S (s_{ij} + \delta_{ij}\sigma)n_j v_i\, dS$$

$$= \int_V (s_{ij} + \delta_{ij}\sigma)_{,j} v_i\, dV + \int_V (s_{ij} + \delta_{ij}\sigma)v_{i,j}\, dV$$

$$= \int_V s_{ij}\tfrac{1}{2}(v_{i,j} + v_{j,i})\, dV = \int_V s_{ij}\mu s_{ij}\, dV = 2k^2 \int_V \mu\, dV,$$

rearranging yields

$$\int_V \mu\, dV = m/(2k^2). \tag{26}$$

The proof is completed by combining (25) and (26). It should be noted that $\max\,\{f(s_{ij}^0) + (\varphi^0)^2\} \geq 0$ because of conditions (18) and (19).

Theorem (15), includes the classical definition of the lower bound, as is seen by taking (18) in the special form

$$f(s_{ij}^0) + (\varphi^0)^2 = 0 \tag{27}$$

In this case $\max\,\{f(s_{ij}^0) + (\varphi^0)^2\}$ vanishes, and (15) reduces to

$$m^0 \leq m. \tag{28}$$

Thus, the new lower bound expressed by the left side of (15) holds for a broader stress field than the statically admissible stress field by taking the integral mean of the yield criterion, (18).

3. Upper bound theorem. The inequality

$$\left(\int_V 2k\mu^* \, dV\right) \max \{\tfrac{1}{2}s_{ij}^*s_{ij}^*\}^{1/2} \geq m \tag{29}$$

holds for any set of v_i^* , s_{ij}^* and μ^* satisfying

$$v_i^* = 0 \qquad \text{on} \quad S_V , \tag{30}$$

$$\delta_{ij}v_{i,j}^* = 0 \qquad \text{in } V, \tag{31}$$

$$\int_{S_T} T_i v_i^* \, dS = 1, \tag{32}$$

$$\mu^* s_{ij}^* = \tfrac{1}{2}(v_{i,j}^* + v_{j,i}^*), \tag{33}$$

$$\int_V \mu^* f(s_{ij}^*) \, dV = 0. \tag{34}$$

Since the right side of (29) is the safety factor, the left side gives an upper bound for the safety factor.

The theorem is proven as follows. It can be shown from (33) that

$$\mu^* = \{\tfrac{1}{2}(v_{i,j}^* + v_{j,i}^*)\tfrac{1}{2}(v_{i,j}^* + v_{j,i}^*)\}^{1/2}/(s_{mn}^*s_{mn}^*)^{1/2} . \tag{35}$$

Also, from (3) and (8),

$$2k \geq (2s_{hl}s_{hl})^{1/2} . \tag{36}$$

Substituting (35) and (36) into $\int_V 2k \, \mu^* \, dV$ we have

$$\int_V 2k\mu^* \, dV \geq \int_V \left[(2s_{hl}s_{hl})^{1/2}\{\tfrac{1}{2}(v_{i,j}^* + v_{j,i}^*)\tfrac{1}{2}(v_{i,j}^* + v_{j,i}^*)\}^{1/2}/(s_{mn}^*s_{mn}^*)^{1/2}\right] dV. \tag{37}$$

Application of Schwarz' inequality and taking the maximum value of $(s_{mn}^*s_{mn}^*)^{1/2}$ lead to

$$\int_V 2k\mu^* \, dV \geq \frac{(2)^{1/2}}{\max\,(s_{mn}^*s_{mn}^*)^{1/2}} \int_V s_{ij}\tfrac{1}{2}(v_{i,j}^* + v_{j,i}^*) \, dV. \tag{38}$$

On the other hand,

$$\int_V s_{ij}\tfrac{1}{2}(v_{i,j}^* + v_{j,i}^*) \, dV = \int_V (s_{ij} + \delta_{ij}\sigma)v_{i,j}^* \, dV = \int_S (s_{ij} + \delta_{ij}\sigma)n_j v_i^* \, dS = m, \tag{39}$$

because of (5), (6), (30) to (32). The proof is completed by combining (38) and (39). It should be mentioned that μ^* and s_{ij}^* are determined from v_i^* by means of (33) and (34).

The theorem, (29), includes the classical definition of the upper bound as is seen by taking (34) in the special form

$$f(s_{ij}^*) = 0 \tag{40}$$

In this case $\max \{\tfrac{1}{2}s_{ij}^*s_{ij}^*\}^{1/2}$ becomes k, and (29) reduces to

$$\int_V 2k^2\mu^* \, dV \geq m. \tag{41}$$

which yields the classical upper bound by taking $(s^*_{mn}s^*_{mn})^{1/2} = (2)^{1/2}k$ in (35). Thus, the new upper bound expressed by the left side of (29) holds for a broader stress field derived from the admissible velocity field in classical limit analysis.

The proposed analysis can be easily extended to include the case with discontinuous stress and velocity fields. The discontinuity of the stress field for the lower bound is admissible if the vector $(s^0_{ij} + \delta_{ij}\sigma^0)n_j$ defined on the surface of discontinuity is continuous. If a velocity field with slip discontinuity is employed for the upper bound, (39) takes the modified form

$$\int_V s_{ij}\tfrac{1}{2}(v^*_{i,j} + v^*_{j,i})\, dV = m + \sum \int_{S_{hk}} T^{(hk)}[v^{*(h)}_T - v^{*(k)}_T]\, dS, \tag{42}$$

where S_{hk} denotes the surface of discontinuity between regions R_h and R_k, $T^{(hk)}$ the tangential stress transmitted across the surface element dS from R_k onto R_h, and $v^{*(h)}_T$ and $v^{*(k)}_T$ the tangential velocity components in R_h and R_k respectively. Thus, it can be shown [3] that (29) becomes

$$\left(\int_V 2k^2\mu^*\, dV\right) \max\, \{\tfrac{1}{2}s^*_{ij}s^*_{ij}\}^{1/2}/k + k \sum \int_{S_{hk}} |v^{*(k)}_T - v^{*(h)}_T|\, dS \geq m. \tag{43}$$

4. Example. The tension specimen subjected to uniformly distributed tensile stress mp per unit length, shown in Figs. 1a and 1b, will be used to illustrate the application of the theorems, assuming a state of plane stress. Due to symmetry only one quarter of the specimen needs be considered.

The assumed stress field consists of zones of constant stresses separated by lines of

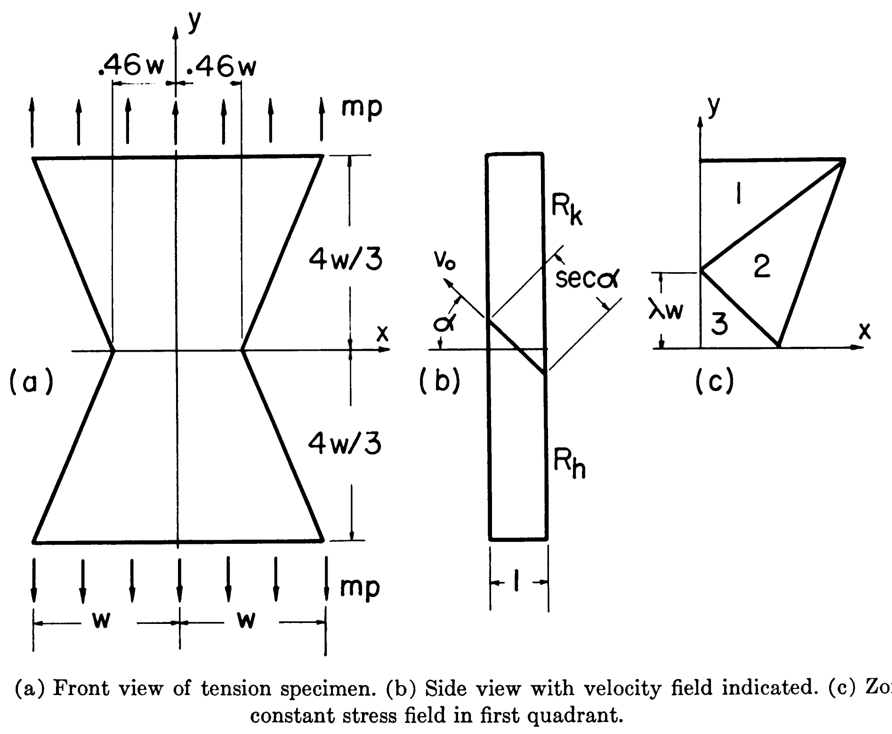

FIG. 1. (a) Front view of tension specimen. (b) Side view with velocity field indicated. (c) Zones of constant stress field in first quadrant.

discontinuity as shown in Fig. 1c. Such a stress field obviously satisfies (16). The boundary conditions and the requirement of continuity of stresses along the lines of discontinuity yield the three non-vanishing stress components σ_x^0, σ_y^0 and τ^0 in the three zones in terms of the redundant parameter λ (Fig. 1c). These stress components are: for zone 1,

$$\sigma_x^0 = (1 - 2A)m^0p, \ \sigma_y^0 = m^0p \quad \text{and} \quad \tau^0 = 0, \tag{44}$$

where

$$A = (1.377 + .417\lambda - .695\lambda^2)/(2.12 + .267\lambda - 1.390\lambda^2);$$

for zone 2,

$$\sigma_x^0 = .281Bm^0p, \qquad \sigma_y^0 = 1.719Bm^0p, \qquad \tau^0 = .695Bm^0p, \tag{45}$$

where

$$B = 1/(.794 + .695\lambda);$$

and for zone 3,

$$\sigma_x^0 = (2.17 - 2C)m^0p, \qquad \sigma_y^0 = 2.17m^0p, \qquad \tau^0 = 0, \tag{46}$$

where

$$C = (.695\lambda^2 + .661\lambda - .1472)/(.639\lambda^2 + .730\lambda).$$

The Mises yield criterion (3) for the case of plane stress [4] reduces to

$$f(\sigma_{ij}) = \tfrac{1}{3}(\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau^2) - k^2 . \tag{47}$$

Substituting (44) to (47) into (22) yields

$$\begin{aligned} F = m^0 &- \int_{S_1} \mu^0\{[4A^2 - 2A + 1](m^0)^2p^2/3 - k^2 + (\varphi^0)^2\}\, dS \\ &- \int_{S_2} \mu^0\{[4.00B^2(m^0)^2p^2/3 - k^2 + (\varphi^0)^2\}\, dS \\ &- \int_{S_3} \mu^0\{[4C^2 - 4.34C + 4.72](m^0)^2p^2/3 - k^2 + (\varphi^0)^2\}\, dS, \end{aligned} \tag{48}$$

where S_1, S_2 and S_3 denote the areas of the three zones. Taking φ^0 and μ^0 to be constants, integrating, and taking the variation of F with respect to μ^0, φ^0 and m^0 yields three simultaneous equations,

$$\partial F/\partial m^0 = 0, \tag{49}$$

$$\partial F/\partial \mu^0 = 0, \tag{50}$$

$$\partial F/\partial \varphi^0 = 0; \tag{51}$$

their solution furnishes

$$m^0 = 3.42wk/[(4A^2 - 2A + 1)S_1 + 4.00B^2S_2 + (4C^2 - 4.34C + 4.72)S_3]^{1/2}p, \tag{52}$$

$$\mu^0 = 1.5/[\{(2A - \tfrac{1}{2})^2 + .75\}S_1 + 4.01B^2S_2 + \{(2C - 1.085)^2 + 3.54\}S_3]m^0p^2 , \tag{53}$$

$$\varphi^0 = 0, \tag{54}$$

where

$$S_1 = (8/3 - 2\lambda)w^2 , \tag{55}$$

$$S_2 = (1.224 + 1.08\lambda)w^2 , \tag{56}$$

$$S_3 = .92\lambda w^2 , \tag{57}$$

w being the half width of the specimen. It should be observed that (53) satisfies (19), and (50) assures that (18) is satisfied. Substitution of (52) on the left of (15) yields a lower bound

$$m' = m'(k/p, \lambda), \tag{58}$$

which is plotted in Fig. 2. It should be noted that (58) defines a piecewise continuous curve. The best bound is $m' = .911\ k/p$ at $\lambda = .386$.

To obtain the lower bound by the classical method, it is necessary to find a stress field that satisfies equilibrium, the boundary conditions and the yield criterion. The stress field given by (44), (45) and (46) is admissible if it satisfies the inequality

$$\sigma_x^2 - \sigma_x\sigma_y + \sigma_y^2 + 3\tau^2 - 3k^2 \leq 0. \tag{59}$$

Substituting (44), (45) and (46) in (59) and rearranging lead to the following inequalities:

$$m^0 \leq 1.732k/[4A^2 - 2A + 1]^{1/2}p, \tag{60}$$

$$m^0 \leq .866k/Bp, \tag{61}$$

$$m^0 \leq 1.732k/[4C^2 - 4.34C + 4.72]^{1/2}p, \tag{62}$$

where A, B and C are as defined previously. The solution of these inequalities is indicated in Fig. 2. The best lower bound is $m^0 = .920\ k/p$ at $\lambda = .387$.

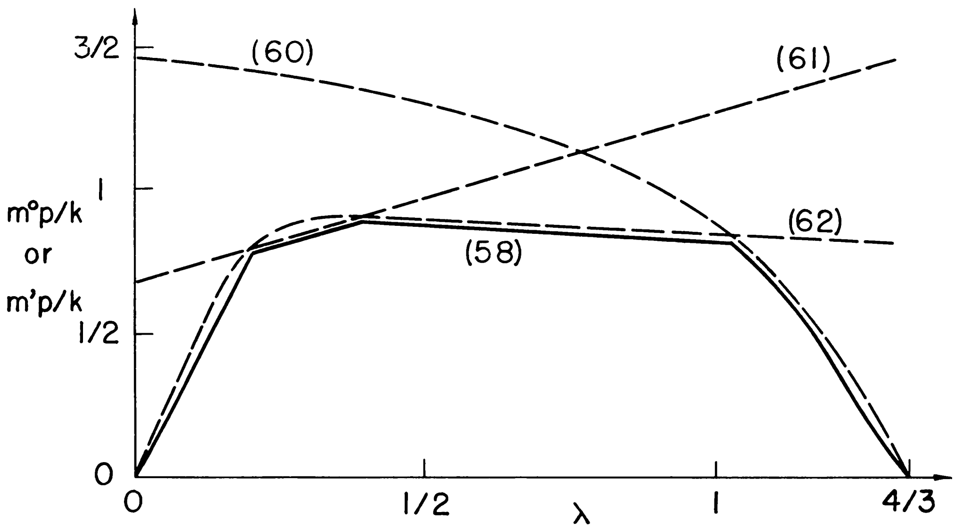

FIG. 2. Lower Bounds Plotted Against Argument λ.

——— Proposed method
— — — Classical method

To evaluate the upper bound of the safety factor, let the upper half of the specimen move with a constant velocity v_0 relative to the lower half as indicated in Fig. 1b. This constant velocity field satisfies (30) and (31), observing that $S_V = 0$ in this case. Substituting this velocity field in (32) furnishes

$$v_0 = 1/(2pw \sin \alpha). \tag{63}$$

Since μ^* vanishes in this case, (43) reduces to

$$k \int_{S_{hk}} |v_T^{*(k)} - v_T^{*(h)}| \, dS \geq m. \tag{64}$$

Integrating the left hand side of (64) with $v_T^{*(k)} = v_0$ and $v_T^{*(h)} = 0$ we have

$$.92kwv_0 \sec \alpha \geq m. \tag{65}$$

Introduction of v_0 from (63) into (65) yields

$$.92 \frac{k}{p} \csc 2\alpha \geq m. \tag{66}$$

Since the left hand side of (66) is minimum when $\alpha = \pi/4$, the best upper bound is $.92\ k/p$.

For this velocity field, equating the internal work to the external work in the classical method yields the same upper bound.

5. Conclusions. The results obtained above show that for this example the upper bounds obtained by the proposed method and the classical method coincide, and the two lower bounds vary similarly with λ (Fig. 2) and are in close agreement.

Drucker [5], in treating a similar problem but using Tresca yield condition, obtained an upper bound which, for the dimensions of the specimen under consideration, is equal to $.92\ k/p$.

If, in the proposed lower bound analysis, μ^0 is assumed to be constant as in the example treated above, the constraint condition (18) is automatically satisfied. If a lower bound solution for the plane strain problem is available, the stress field can be conveniently used to obtain the lower bound of the plane stress problem by means of (15), subject only to constraint condition (19) which is easily satisfied. For example, with the stress field used by Prager and Hodge [6] in a plane strain problem involving a tension specimen with semi-circular notches, (15) yields a lower bound for the problem under consideration equal to $.863\ k/p$ which is satisfactory.

In problems where the stress field or the velocity field is a function of the space coordinates, the proposed method has an advantage over the classical method in that (18) or (34), the integral mean of the yield condition, eliminates the space variables.

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