—NOTES—

DIFFERENCE METHODS FOR A NON LINEAR ELLIPTIC SYSTEM OF PARTIAL DIFFERENTIAL EQUATIONS*

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Abstract. This paper proves the existence and uniqueness of non-negative solutions to the Dirichlet problem associated with the nonlinear elliptic system

$$\Delta u_k = b_k \prod_{l=1}^{m} u_l^{n(l)}, \qquad k = 1, \cdots, m \tag{*}$$

where the b_k and the Dirichlet data $u_k = \varphi_k$ are non-negative.

An iteration scheme is proposed for solving the difference equations associated with (*) and a bound on the error between any iteration and the solutions to the difference equations is established.

1. Introduction. In [1], Ablow and Perry present iterative methods for solving the Dirichlet problem—over a plane region Ω—associated with the equation $\Delta u = u^2$. They were motivated, in part, by a desire to solve the Dirichlet problem for the equation

$$\Delta u_k = b_k \prod_{l=1}^{m} u_l^{n(l)}, \qquad k = 1, \cdots, m, \tag{1}$$

with Dirichlet data $u_k = \varphi_k$. Here, b_k are non-negative real numbers, $n(l)$ positive integers, and $n \equiv \sum_{l=1}^{m} n(l)$. This equation gives the concentration u_i of the i-th labile species in an n-th order, diffusion supported, chemical reaction.

The present paper proves the existence of a unique positive solution of (1) and presents a discretization of an extension of the iterative methods posed in [1]. We shall also discuss some difference methods for the problem

$$\Delta u = u^{2n} \quad \text{in} \quad \Omega, \quad \text{with} \quad u = \varphi \quad \text{on} \quad \partial\Omega, \tag{2}$$

which is a special case of (1).

2. The continuous problem. In this section we shall prove existence and uniqueness of a positive solution of (1). [Note that equation (2) is a special case of (1)]. For convenience of notation we shall assume that $m = 3$ in the proof.

Theorem 1. If $b_k \geq 0$ and $\varphi_k \geq 0$, then there exists one and only one non-negative solution of (1). Moreover,

$$\max_{\Omega} |u_k| \leq \max_{\partial\Omega} |\varphi_k|.$$

Proof: (*Uniqueness*) Let u_1, u_2, u_3 and v_1, v_2, v_3 be two distinct sets of solutions to (1) with u_k and v_k non-negative. Then

*Received Oct. 21, 1964; revised manuscript received Dec. 28, 1964. Sponsored by the Mathematics Research Center, United States Army, Madison Wisconsin, under Contract No.: DA-11-022-ORD-2059, and by the U. S. Army, Ballistic Research Laboratories, Aberdeen Proving Ground, Maryland.

$$\Delta(u_k - v_k) = b_k[u_2^{n(2)}u_3^{n(3)}P_{n(1)-1}(u_1, v_1)(u_1 - v_1) + v_1^{n(1)}v_3^{n(3)}P_{n(2)-1}(u_2, v_2)(u_2 - v_2) + v_1^{n(1)}u_2^{n(2)}P_{n(3)-1}(u_3, v_?)(u_3 - v_3)], \tag{3}$$

where $P_m(\xi, \eta) = \sum_{\alpha=1}^{m} \xi^{m-\alpha}\eta^{\alpha}$. We observe that all the coefficients of $(u_k - v_k)$ in the right-hand side of (3) are non-negative.

Let $\Omega_k^+ = \{(x, y) : u_k(x, y) - v_k(x, y) > 0\}$ and $\Omega_{123}^+ = \{(x, y)$: right-hand side of (3) $> 0\}$; in a similar manner—using strict inequality—we define Ω_k^- and Ω_{123}^-. We see that $\Omega_k^+ \cap \Omega_k^- = \phi$, $\partial\Omega_k^+ = \partial\Omega_k^-$, etc.

Since $u_k - v_k = 0$ iff $u_l - v_l = 0$, then the Maximum Principle ([2], p. 326) shows that $(x, y) \in \Omega_k^- \cup \partial\Omega_k^-$ for all k. Hence $\Omega_{123}^+ \subset \bigcap_k (\Omega_k^- \cup \partial\Omega_k^-)$. By analogous reasoning $\Omega_{123}^- \subset \bigcap_k (\Omega_+^k \cup \partial\Omega_k^+)$.

If $(x, y) \in \bigcap_k(\Omega_k^+ \cup \partial\Omega_k^+)$, then $(x, y) \in \Omega_{123}^+$. Hence,

$$\Omega_{123}^- \subset \bigcap_k (\Omega_k^+ \cup \partial\Omega_k^+) \subset \Omega_{123}^+ .$$

Since $\Omega_{123}^- \cap \Omega_{123}^+ = \phi$, $\Omega_{123}^- = \phi$. Change in the sign of the above shows that $\Omega_{123}^+ = \phi$. Thus $u_k - v_k \equiv 0$ for all k and we have a contradiction on the distinctness of the elements u_1, u_2, u_3 and v_1, v_2, v_3.

The estimate $\max_\Omega |u_k| \le \max_{\partial\Omega} |\varphi_k|$ is established by applying the Maximum Principle to the equation

$$\Delta(u_k - h_k) = b_k u_1^{n(1)}u_2^{n(2)}u_3^{n(3)},$$

where h_k is the harmonic function taking on the values φ_k on $\partial\Omega$.

(Existence) Let us denote by ${}_kC_l^+(\Omega)$ the set of functions that are continuous and non-negative on $\bar{\Omega}$, are equal to φ_k on $\partial\Omega$, and have l continuous derivatives in Ω.

Let B_k denote the subset of ${}_kC_0^+$ such that its elements ξ satisfies $\max_\Omega |\xi| \le \max_{\partial\Omega} |\varphi_k|$. Let $\xi_k \in {}_kC_0^+$. Let $\Phi_k(\xi_1, \cdots, \xi_3)$ be *the* solutions to the equations, with $\Phi_k = \varphi_k$ on $\partial\Omega$,

$$\Delta\Phi_k(\xi_1, \cdots, \xi_3) = b_k\xi_i^{n(i)}\xi_j^{n(j)}\xi_k^{n(k)-1}\Phi_k, \tag{4}$$

where the Laplace operator Δ is taken with respect to the independent variables x and y and the indicies i and j are distinct from k. Then, $\max_\Omega |\Phi_k(\xi_1, \cdots, \xi_3)| \le \max_{\partial\Omega} |\varphi_k|$. Therefore Φ_k is defined on B_k and maps B_k into itself. Moreover, Φ_k has Hoelder continuous second derivatives.

The set B_k is a convex, closed and compact subset of ${}_kC_0^+$. The mapping Φ_k is a continuous function of ξ_1, ξ_2, ξ_3. The Schauder Fixed-Point Theorem (Courant and Hilbert [2], p. 357) establishes the existence of at least one solution to (1).

3. Approximate difference equations to (2). Place a square grid on Ω with grid width h. The grid points are expressible in the form (mh, nh) with m and n taking on integer values.

Let $P_i \equiv (x_0, y_0)$ be a grid point. The *neighbors* of P_i, denoted by $P_{i\nu}(\nu = 1, \cdots, 4)$, are the points of the form $(x + h, y)$, $(x - h, y)$, $(x, y + h)$, and $(x, y - h)$. Let Ω_h consist of those grid points each of which has all of its neighbors in Ω. Let $\partial\Omega_h$ denote the boundary of Ω_h, i.e. those grid points in Ω which have at least one neighbor outside Ω.

We place a lexicographic ordering on the points of Ω_h and denote these points by $P_1, \cdots, P_N$. The points of $\partial\Omega_h$ are denoted by $P_{N+1}, \cdots, P_M$.

For a function $f(x, y)$ defined on Ω, we denote by f_i the value of f at P_i. The quantity $\|\cdot\|$ denotes the maximum of the absolute values of the components of a vector.

The continuous iterative methods presented in Ablow and Perry [1] may be written in difference form, using the five-point approximation, as:

$$\Delta_h U_{(m)} = u^{2n}_{(m-1)}, \qquad \Delta_h U'_{(m)} - u^{2n-1}_{(m-1)} U'_{(m)} = 0, \tag{5}$$

where $\Delta_h f_i \equiv h^{-2} \{\sum_\nu f_{i\nu} - 4f_i\}$, U is used to represent a vector, u represents the continuous solution and $'$ distinguishes between the vector solution of (5.1) and (5.2).

The above equations assume that the function $u_{(m-1)}$ is known. If this were the case, there would be little need for the difference equation. Hence, we replace $u_{(m-1)}$ by its "approximate" $U_{(m-1)}$ or $U'_{(m-1)}$; the resulting equation we will call an *approximate difference equation.*

The approximate difference equations may be written in matrix form as

$$LU_{(m)} = U^{2n}_{(m-1)} + \sigma_{(m)}\varphi, \qquad L'_{(m-1)} U'_{(m)} = \sigma'_{(m)}\varphi, \tag{6}$$

where the Laplace matrices L, $L'_{(m-1)}$ have as their (k, l)—entry the coefficient of $U_{(m)l}$, $U'_{(m)l}$—the subscript l denotes the l-th component of the vector $U_{(m)}$—when (5) is evaluated at $U_{(m)k}$, $U'_{(m)k}$ and the boundary matrices $\sigma_{(m)}$, $\sigma'_{(m)}$ have as their (l, j)—entry the coefficient of the j-th boundary point when (5) is evaluated at $U_{(m)l}$, $U'_{(m)l}$.

LEMMA. If $\varphi \geq 0$, the following are true:

a) $\sigma_{(m)} = \sigma'_{(m)} \equiv \sigma < 0$ for all m.
b) If $D(\xi)$ denotes the diagonal matrix having its (i, i)—component given by ξ_i, then $L'_{(m)} = L—D(U'^{2n-1})$.
c) $-L$ is monotone.
d) Let $G = L^{-1}$ and $\Gamma^{ij} \equiv \sum_s G^{is}\sigma^{sj}$. Then, for all i, $\sum_j \Gamma^{ij} = 1$.
e) If $\varphi \geq \epsilon > 0$ and $8\epsilon \geq ||\varphi||^{2n} d^2$, then $U_{(m)} \leq U_{(1)}$ and $U_{(m)} \geq 0$.
f) $-L'_{(m)}$ is monotone, $U'_{(m)} \geq 0$, and if $G'_{(m)} = L'^{-1}_{(m)}$, then $0 \leq \sum_s - G'^{is}_{(m)} \leq \sum_s - G^{is}$.

PROOF. a) and b) follow from the definitions of $\sigma_{(m)}$, $\sigma'_{(m)}$, and $L'_{(m)}$

c) The row-sum criterion, sign distribution, and irreducibility hypothesis of Collatz ([3], p. 45) are satisfied.

d) Let $V_i \equiv \sum_j \Gamma^{ij}$. Then V_i satisfies the difference problem $\Delta_h V_i = 0$ for $i = 1, \cdots, N$ with $V_j = 1$ for $j = N + 1, \cdots, M$. From the uniqueness of Laplace's difference problem we conclude that $V_i \equiv 1$ for all i.

e) Equation (6) shows that

$$U_{(m)} = \sum_{s=1}^{N} G^{is} U^{2n}_{(m-1)s} + \sum_{j=N+1}^{M} \Gamma^{ij}\varphi_j . \tag{7}$$

For $m = 1$, $0 < \epsilon \leq U_{(1)} \leq ||\varphi||$. If we assume that one component of Ω, say the x-component, lies in the strip $0 \leq x \leq d$, then (Bers [4], p. 231) $0 \leq \sum_s - G^{is} \leq d^2/8$. Therefore,

$$U_{(2)} \geq -d^2\, ||\varphi||^{2n}/8 + \epsilon \sum_{j=N+1}^{M} \Gamma^{ij} \geq 0.$$

From the equation $\Delta_h(U_{(2)} - U_{(1)}) = U^{2n}_{(1)}$, we conclude—since $U_{(2)} - U_{(1)} = 0$ on

$\partial\Omega_h$—that $U_{(3)} \geq U_{(2)} \geq 0$. The equation $\Delta_h(U_{(3)} - U_{(1)}) = U_{(2)}^{2n}$ shows that $U_{(3)} \geq 0$. Induction produces the desired result.

f) Again we have that $U'_{(1)} \geq 0$. Hence, reasoning as in c), $-L'_{(2)}$ is monotone. Since $U'_{(2)} = L'^{-1}_{(1)}\sigma\varphi \geq 0$, $-L'_{(3)}$ is monotone. The first result now follows by induction.

Let η be the N-component vector with each entry one. Then

$$(-L'_{(m)})(-L'_{(m)})^{-1}\eta = \eta \leq [I + D[U'^{2n-1}_{(m-1)}]\cdot(-L)^{-1}]\eta = (-L'_{(m)})(-L^{-1})\eta.$$

Therefore, $\sum_s - G'^{is}_{(m)} \leq \sum_s - G^{is}$.

We shall now show that the sequences $\{U_{(m)}\}$ and $\{U'_{(m)}\}$ converge.

THEOREM 2. a) If $\varphi \geq \epsilon > 0$ and $8\epsilon \geq d^2\, ||\varphi||^{2n}$, then $\{U_{(m)}\}$ is a convergent sequence

b) If $\varphi \geq 0$, then $\{U'_{(m)}\}$ is a convergent sequence.

PROOF. a) The vector $U_{(m+1)} - U_{(m-1)} = 0$ on $\partial\Omega_h$ and satisfies in Ω_h the equation

$$\Delta_h(U_{(m+1)} - U_{(m-1)}) = (U_{(m)} - U_{(m-2)})P_{2n-1}(U_{(m)}\ ,\ U_{(m-2)}).$$

If m is odd, $U_{(m)} - U_{(m-2)} \leq 0$. Since $U_{(m)}$ and $U_{(m-2)}$ are non-negative, $U_{(m+1)} - U_{(m-1)} \geq 0$. Hence, $\{U_{(2m)}\}$ is a non-decreasing sequence and $\{U_{(2m+1)}\}$ is a non-increasing sequence. Hence, both converge to solutions of the equation $\Delta_h U = U^{2n}$. Once we show the uniqueness of a solution to this equation we have the equality of the limits.

Let V and W denote the limits of $\{U_{(2m)}\}$ and $\{U_{(2m+1)}\}$ respectively. Then $V \leq W$. Therefore, the equation $\Delta_h(W - V) = (W - V)P_{2n-1}(W, V)$ implies that $W \leq V$, i.e. $V = W$.

b) The vector $W'_{(m)} \equiv U'_{(m)} - U'_{(m-1)} = 0$ on $\partial\Omega_h$ and satisfies the equation

$$\Delta_h W'_{(m)} - U'^{2n-1}_{(m-1)}W'_{(m)} = U'_{(m-1)}W'_{(m-1)}P_{2n-2}(U'_{(m-1)}\ ,\ U'_{(m-2)}).$$

Since $U'_{(m-1)}$ and $U'_{(m-2)}$ are non-negative, $W'_{(m)}$ has sign opposite to $W'_{(m-1)}$ with $W'_{(1)} \geq 0$. The vector $V'_{(m)} \equiv U'_{(m)} - U'_{(m-2)}$ is zero on $\partial\Omega_h$ and satisfies the equation

$$\Delta_h V'_{(m)} - U'^{2n-1}_{(m-3)}V'_{(m)} = U'_{(m)}V'_{(m-1)}P_{2n-2}(U'_{(m-1)}\ ,\ U'_{(m-3)}).$$

Since $U'_{(2)} \geq 0$, we conclude that $\{U'_{(2m)}\}$ is a non-decreasing sequence and $\{U'_{(2m+1)}\}$ is a non-increasing sequence. Hence, they both converge to a solution of the difference equation $\Delta_h U = U^{2n}$ with

$$U'_{(2m)} \leq U \leq U'_{(2m+1)}\ .$$

REMARKS. a) If we considered the equation $\Delta u = bu^{2n}$ with $b \geq 0$, then the condition needed to prove that $U_{(m)} \geq 0$ becomes $8\epsilon \geq bd^2\, ||\phi||^{2n}$.

b) Greenspan [5] has discretized an iterative scheme posed by Pohozaev [6] of the form $\Delta u_{(m)} - 2u_{(m-1)}u_{(m)} = -u^2_{(m-1)}$ which leads to a *monotonically* decreasing sequence $\{u_{(m)}\}$. The associated approximate difference equations have a sequence of solutions, say $\{V_{(m)}\}$, that converges monotonically to a solution of the difference equation $\Delta_h V = V^2$. However, one is not able to obtain an estimate for the $||V - V_{(m)}||$. The method presented in this paper shows that $||U - U_{(m)}|| \leq ||U_{(m)} - U_{(m-1)}||$. This is a definite advantage.

c) The convergence of the solution of the difference equations to the differential equation proceeds as in Bers [4].

4. An approximate difference equation for (2). Let ${}_{(m)}U_k$ be an M-component vector which satisfies the following approximate difference problem:

$$\Delta_h\ {}_{(m)}U_{k,i} - b_k \cdot {}_{(m-1)}U_{l,i}^{n(l)} \cdot {}_{(m-1)}U_{p,i}^{n(p)} \cdot {}_{(m-1)}U_{k,i}^{n(k)-1} \cdot {}_{(m)}U_{k,i} = 0 \qquad i = 1, \cdots, N$$
$${}_{(m)}U_{k,j} = \varphi_i ; \qquad j = N+1, \cdots, M; \tag{8}$$

here k, l, and p take on distinct integer values between one and three. We set ${}_0U_{k,i} \equiv 0$ for all k and i.

We first observe that ${}_{(m)}U_{k,i} \geq 0$ for all i, m, and k and that the Laplacian matrix corresponding to (8), denoted by ${}_{(m)}L_k$, has all elements of its inverse non-positive.

Let ${}_{(m)}W_k \equiv {}_{(m)}U_k - {}_{(m-1)}U_k$. Then ${}_{(m)}W_k = 0$ on $\partial\Omega_h$ and satisfies the equation

$$\begin{aligned}\Delta_h\ {}_{(m)}W_k - b_k\ {}_{(m-1)}U_l^{n(l)}\ {}_{(m-1)}U_p^{n(p)}\ {}_{(m-1)}U_k^{n(k)-1}\ {}_{(m)}W_k \\ = b_k\{{}_{(m-1)}U_k[{}_{(m-2)}U_l^{n(l)}\ {}_{(m-2)}U_p^{n(p)}({}_{(m-1)}U_k^{n(k)-1} - {}_{(m-2)}U_k^{n(k)-1}) \\ + {}_{(m-1)}U_k^{n(k)-1}[{}_{(m-2)}U_l^{n(l)}({}_{(m-1)}U_p^{n(p)} - {}_{(m-2)}U_p^{n(p)} \\ + {}_{(m-1)}U_p^{n(p)}({}_{(m-1)}U_l^{n(l)} - {}_{(m-2)}U_l^{n(l)})]]\} = 0.\end{aligned} \tag{9}$$

Since ${}_1U_k \geq {}_0U_k \equiv 0$, we conclude that, for all k, ${}_{(m)}W_k$ has sign opposite to ${}_{(m-1)}W_k$.

Let ${}_{(m)}W_k' = {}_{(m+1)}U_k - {}_{(m-1)}U_k$. Then ${}_{(m)}W_k'$ satisfies the equation

$$\begin{aligned}\Delta_h\ {}_{(m)}W_k - b_k\ {}_{(m-1)}U_l^{n(l)}\ {}_{(m-1)}U_p^{n(p)}\ {}_{(m-1)}U_k^{n(k-1)}\ {}_{(m)}W_k' \\ = b_k\{{}_{(m-1)}U_k[{}_{(m-2)}U_l^{n(l)}\ {}_{(m-2)}U_p^{n(p)}({}_{(m)}U_k^{n(k)-1} - {}_{(m-2)}U_k^{n(k)-1}) + {}_{(m)}U_k^{n(k)-1} \\ \cdot[{}_{(m-2)}U_l^{n(l)}({}_{(m)}U_p^{n(p)} - {}_{(m-2)}U_p^{n(p)}) + {}_{(m-1)}U_p^{n(p)}({}_{(m-1)}U_l^{n(l)} - {}_{(m-2)}U_l^{n(l)})]]\} = 0\end{aligned}$$

Reasoning as in §3 shows that $\{{}_{(2m)}U_k\}$ is a non-decreasing sequence and $\{{}_{(2m+1)}U_k\}$ is a non-increasing sequence. Let us denote their respective limit, in m, by μ and v. The identity of μ and v now follows by the same arguments as in §3.

We have therefore exhibited a difference method for (2) that converges uniformly to the solution of the system of differential equations. It should be pointed out that this method does allow for an excellent error estimate in the solution of the system of approximate difference equations.

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