A FREE BOUNDARY PROBLEM FOR THE HEAT EQUATION WITH HEAT INPUT AT A MELTING INTERFACE*

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1. Statement of Problem. We consider a slab of heat conducting material initially occupying $0 \leq x \leq a$; the slab is insulated at $x = 0$ and there is heat input $Q(t)$ per unit area per unit time at the opposite face. If melted material is formed we assume it is removed immediately; the heat input is always at the melting interface. This problem has been considered by Landau [1], Boley [2; 3; 4] and Citron [5]. The problem is formulated as follows: we wish to find the equation $x = s(t)$ of the melting interface and the temperature $u(x, t)$ in the slab when these quantities are subject to

$$\begin{gathered} \kappa u_{xx} = u_t\,, \quad u(x, 0) = f(x), \quad u_x(0, t) = 0, \\ u(s(t), t) = 0, \quad -\rho l s'(t) + k u_x(s(t), t) = Q(t), \quad s(0) = a, \end{gathered} \tag{1}$$

where $f(x)$ is the initial temperature distribution (we assume $f(x) \leq 0$, $f(a) = 0$), k is the thermal conductivity, κ the thermal diffusivity, ρ the density, l the latent heat of fusion, u the temperature, and the melting temperature has been taken to be 0.

The formulation (1) is not completely precise for the following reason: if the heat flux $Q(t)$ is insufficient to maintain melting for some positive time interval then we must drop the condition $u(s(t), t) = 0$ for that time and deal with the conventional problem in which one face is insulated and the other face is fixed and has a prescribed heat flux. We note that there are two possibilities.

(a) The slab melts without pause, i.e., $s'(t) \leq 0$ with $s'(t) = 0$ at isolated times t; it may or may not melt completely in a finite time.

(b) There are intermittent periods of non-melting; in this case, as remarked above, we must drop the condition $u(s(t), t) = 0$ during the non-melting periods. Complete melting may or may not occur in a finite time.

The strongest theorem we could state would specify which type of situation exists given $Q(t)$ and $f(t)$, i.e. (a) or (b), and would assert the existence and uniqueness of a solution. No theorem of this kind seems to have been proved for (1) although a uniqueness theorem has been given by Boley [4, p. 5]. Existence theorems for various kinds of free boundary problems for the heat equation have been proved by Evans [6], Miranker [7], Kyner [8; 9], Friedman [10; 11], Trench [12], Douglas and Gallie [13], and others. In the problem treated by these authors the melted part remains intact and the heat is applied at the fixed end of the melted material. This leads to a simpler boundary condition at the melting face.

In this paper we use a modification of Miranker's method to prove an existence and uniqueness theorem for (1) which may be stated as follows.

*Received December 17, 1964; revised manuscript received May 10, 1965. This work was done when the author was a member of the technical staff at Space Technology Laboratories, Redondo Beach, California.

Theorem. Suppose $f'(x)$ exists and is continuous in some interval $\delta \leq x \leq a$. Let $f(x)$ be continuous and bounded in $0 < x \leq a$ and let $Q(t) \geq 0$ be continuous for $t \geq 0$. Then if

$$\rho l s'(0) = k u_x(s(0), 0) - Q(0) = k f'(a) - Q(0) < 0 \tag{2}$$

there exists T, $u(x, t)$, and $s(t)$ satisfying (1) for $0 \leq x \leq s(t)$, $0 \leq t \leq T$, such that $s'(t)$ is continuous and negative; the solution is unique.

We note that this is a local theorem, asserting that if melting has begun then it will continue for some time thereafter. Before proceeding to the proof of the theorem we observe that complete melting in a finite time t occurs if and only if

$$\int_0^t Q(\tau)\, d\tau = \rho l a - \frac{k}{\kappa} \int_0^a f(x)\, dx. \tag{3}$$

Eq. (3) follows directly from (1); a derivation can be found in the book of Boley and Weiner [2, p. 192].

2. Proof of theorem. The method used by Miranker is based on a method devised by Kolodner [14] for a class of free boundary problems for the heat equation. Kolodner derived a number of non-linear integral equations ([14], p. 13, Eqs. (5.1), (5.2), (5.3)) each of which is satisfied by the unknown boundary.[1] Miranker applied one of these integral equations to his problem and proved the existence of a solution by applying the principle of contracting mappings, i.e., a contracting mapping of a complete metric space into itself has a unique fixed point.

For the purpose of proving the existence of a solution of (1) the method of Kolodner does not work, since his equations (5.1) and (5.3) of [14] turn out to be integral equations of the first kind and therefore unsuitable for a fixed-point argument, while his Eq. (5.2) is applicable only if $s'(t) \geq 0$. But in (1) we clearly have $s'(t) \leq 0$. We may, however, proceed by taking, as a starting point, the discussion of Goursat [15, p. 316].

In order to simplify the notation suppose units have been chosen so that $\kappa = 1$ and write $\alpha = \rho l$ and $\beta = k$. We represent $u(x, t)$ as follows:

$$u(x, t) = \int_0^t \frac{\mu(\tau)}{2[\pi(t - \tau)]^{1/2}} \left[\varphi\left(\frac{(x - s(\tau))^2}{4(t - \tau)}\right) + \varphi\left(\frac{(x + s(\tau))^2}{4(t - \tau)}\right) \right] d\tau + \int_0^a \frac{f(\xi)}{2(\pi t)^{1/2}} \left[\varphi\left(\frac{(x - \xi)^2}{4t}\right) + \varphi\left(\frac{(x + \xi)^2}{4t}\right) \right] d\xi, \tag{4}$$

where $\varphi(z) = e^{-z}$ and $\mu(t)$ is a continuous function to be determined. The function $u(x, t)$ is an even solution of the heat equation such that $u(x, 0+) = f(x)$ for $0 < x \leq a$; since it is even, $u_x(0, t) = 0$. From (4) we get

$$u_x(x, t) = -\int_0^t \frac{\mu(\tau)}{4[\pi(t - \tau)^3]^{1/2}} \left[(x - s(\tau))\varphi\left(\frac{(x - s(\tau))^2}{4(t - \tau)}\right) + (x + s(\tau))\varphi\left(\frac{(x + s(\tau))^2}{4(t - \tau)}\right) \right] d\tau - \int_0^a \frac{f(\xi)}{4(\pi t^3)^{1/2}} \left[(x - \xi)\varphi\left(\frac{(x - \xi)^2}{4t}\right) + (x + \xi)\varphi\left(\frac{(x + \xi)^2}{4t}\right) \right] d\xi.$$

We now let x approach $s(t)$ from the left. Then ([15], p. 308)

[1] The geometry in Kolodner's paper is such that, to be consistent with his notation, we would have to take the insulated face at $x = a$ and apply the heat at the opposite face.

$$\frac{\alpha}{\beta} s'(t) + \frac{1}{\beta} Q(t) = \frac{1}{2} \mu(t) - \int_0^t \frac{\mu(\tau)}{4[\pi(t-\tau)^3]^{1/2}} \left[(s(t) - s(\tau))\varphi\left(\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\right) + (s(t) + s(\tau))\varphi\left(\frac{(s(t)+s(\tau))^2}{4(t-\tau)}\right) \right] d\tau - \int_0^a \frac{f(\xi)}{4(\pi t^3)^{1/2}} \left[(s(t) - \xi)\varphi\left(\frac{(s(t)-\xi)^2}{4t}\right) + (s(t) + \xi)\varphi\left(\frac{(s(t)+\xi)^2}{4t}\right) \right] d\xi. \tag{5}$$

Eq. (5) is one equation for the two unknowns $s(t)$ and $\mu(t)$. Another equation is obtained by letting $x \to s(t)$ in (4). Then

$$0 = \int_0^t \frac{\mu(\tau)}{2[\pi(t-\tau)]^{1/2}} \left[\varphi\left(\frac{(s(t)-s(\tau))^2}{4(t-\tau)}\right) + \varphi\left(\frac{(s(t)+s(\tau))^2}{4(t-\tau)}\right) \right] d\tau + \int_0^a \frac{f(\xi)}{2(\pi t)^{1/2}} \left[\varphi\left(\frac{(s(t)-\xi)^2}{4t}\right) + \varphi\left(\frac{(s(t)+\xi)^2}{4t}\right) \right] d\xi.$$

This integral equation of the first kind may be converted into an equation of the second kind ([15], p. 340):

$$\frac{\pi^{1/2}}{2} \mu(t) + \int_0^t \frac{\partial K(t,\tau)}{\partial t} \mu(\tau)\, d\tau = -\frac{d}{dt} \int_0^t \frac{1}{(t-\eta)^{1/2}} \left[\int_0^a \frac{f(\xi)}{(\pi\eta)^{1/2}} \left\{ \varphi\left(\frac{(s(\eta)-\xi)^2}{4\eta}\right) + \varphi\left(\frac{(s(\eta)+\xi)^2}{4\eta}\right) \right\} d\xi \right] d\eta, \tag{6}$$

where

$$K(t,\tau) = \int_\tau^t \frac{1}{2[\pi(\eta-\tau)(t-\eta)]^{1/2}} \left[\varphi\left(\frac{(s(\eta)-s(\tau))^2}{4(\eta-\tau)}\right) + \varphi\left(\frac{(s(\eta)+s(\tau))^2}{4(\eta-\tau)}\right) \right] d\eta. \tag{7}$$

Eqs. (5) and (6) constitute two equations for the two unknowns $\mu(t)$ and $s(t)$; we will show that these two equations, together with the initial condition $s(0) = a$, have over some interval $0 \le t \le T$, continuous solutions $\mu(t)$ and $s(t)$ such that $s'(t)$ exists and is continuous.

The derivatives which appear in Eq. (6) may be evaluated; the details are carried out in the Appendix. Eq. (6) may then be written

$$\mu(t) = -\frac{2}{\pi^{1/2}} \int_0^t \int_\tau^t \frac{\psi_\eta(\eta,\tau)}{t-\tau} \left(\frac{\eta-\tau}{t-\eta}\right)^{1/2} \mu(\tau)\, d\eta\, d\tau - \frac{2}{\pi^{1/2}} \int_0^t \frac{g'(\eta)}{(t-\eta)^{1/2}}\, d\eta, \tag{8}$$

where

$$\psi(\eta,\tau) = \frac{1}{2\pi^{1/2}} \left[\varphi\left(\frac{(s(\eta)-s(\tau))^2}{4(\eta-\tau)}\right) + \varphi\left(\frac{(s(\eta)+s(\tau))^2}{4(\eta-\tau)}\right) \right],$$
$$g(\eta) = \int_0^a \frac{f(\xi)}{(\pi\eta)^{1/2}} \left[\varphi\left(\frac{(s(\eta)-\xi)^2}{4\eta}\right) + \varphi\left(\frac{(s(\eta)+\xi)^2}{4\eta}\right) \right] d\xi. \tag{9}$$

If we assume that (5) and (8) have solutions such that $\mu(t)$ and $s'(t)$ are continuous in $0 \le t \le T$ then we may let $t \to 0$ in these equations and obtain the limits

$$\mu(0) = f'(a), \qquad \frac{\alpha}{\beta} s'(0) + \frac{1}{\beta} Q(0) = f'(a). \tag{10}$$

The details of these calculations are carried out in the Appendix. We write (5) and (8) in the form

$$\frac{\alpha}{\beta} s'(t) + \frac{1}{\beta} Q(t) - \tfrac{1}{2}\mu(t) = F(s(t), \mu(t)), \qquad \mu(t) = G(s(t), \mu(t)). \tag{11}$$

We define now the Banach space of vectors $(s(t), \mu(t))$, where $\mu(t)$ and $s(t)$ are continuous functions on $0 \leq t \leq T$, T to be determined, and such that $s'(t)$ exists and is continuous; the norm for the Banach space is taken to be

$$||(s, \mu)|| = |s(0)| + ||s|| + ||\mu||, \tag{12}$$

where

$$||s|| = \max_{0 \leq t \leq T} |s'(t)|, \qquad ||\mu|| = \max_{0 \leq t \leq T} |\mu(t)|.$$

We select two positive constants ζ_1 and ζ_2, with $\zeta_1 < -s'(0)$, and define a closed subset S of the Banach space by prescribing that $(s, \mu) \varepsilon S$ is and only if

$$\begin{aligned} &\text{(a)} \qquad s(0) = a, \\ &\text{(b)} \qquad \mu(0) = f'(a), \qquad \frac{\alpha}{\beta} s'(0) + \frac{1}{\beta} Q(0) = f'(a), \\ &\text{(c)} \qquad |s'(t) - s'(0)| \leq \zeta_1, \qquad |\mu(t) - f'(a)| \leq \zeta_2. \end{aligned} \tag{13}$$

It will be convenient to write $A = \zeta_1 + s'(0)$, $B = -\zeta_1 + s'(0)$. Thus we may write the first inequality of (13c) as

$$B \leq s'(t) \leq A < 0. \tag{14}$$

We may now state the existence and uniqueness theorem: consider the mapping

$$\frac{\alpha}{\beta} s^{*\prime}(t) + \frac{1}{\beta} Q(t) - \tfrac{1}{2}\mu^*(t) = F(s(t), \mu(t)), \qquad s^*(0) = a, \qquad \mu^*(t) = G(s(t), \mu(t)). \tag{15}$$

Then T may be chosen so that (15) is a contracting mapping of S into itself, i.e.,

$$||(s_2^* - s_1^*, \mu_2^* - \mu_1^*)|| \leq c(T, \zeta_1, \zeta_2)\, ||(s_2 - s_1, \mu_2 - \mu_1)||,$$

where $0 < c(T, \zeta_1, \zeta_2) < 1$. Thus we may conclude that the mapping (15) has a fixed point, i.e., that (11) has a unique solution.

The theorem stated above is a local existence theorem. In order to state a theorem valid for all $t > 0$ which would perhaps exclude the case of intermittent melting it would probably be sufficient to take $Q(t)$ in excess of a positive constant and $|f'(t)|$ less than a positive constant. Complete melting would then occur in a finite time. These conditions are, of course, unduly stringent.

Appendix

In the Appendix we evaluate first the derivatives that appear in (6). In the second section we prove that the mapping (15) is a mapping of the Banach space of vectors $(s(t), \mu(t))$ into itself, i.e., that $s^{*\prime}(t)$ and $\mu^*(t)$ are also continuous. In the third section we prove that the set S, defined by (13), is mapped into itself. Finally, in the fourth section it is shown that the mapping of S into itself is contracting.

1. Evaluation of the derivatives in (6). We show first that

$$\frac{d}{dt}\int_0^t \frac{g(\eta)}{(t-\eta)^{1/2}}\,d\eta = \int_0^t \frac{g'(\eta)}{(t-\eta)^{1/2}}\,d\eta, \tag{16}$$

where $g(\eta)$ is given by (9). If $s'(\eta)$ exists and is continuous then $g'(\eta)$ exists and is continuous for $\eta > 0$; it may be obtained by differentiating under the integral in (9):

$$\begin{aligned} g'(\eta) = \int_0^a \Bigg\{ &-\frac{f(\xi)}{4(\pi\eta^3)^{1/2}}\left[\varphi\left(\frac{(s(\eta)-\xi)^2}{4\eta}\right)+\varphi\left(\frac{(s(\eta)+\xi)^2}{4\eta}\right)\right] \\ &+\frac{f(\xi)}{8(\pi\eta^5)^{1/2}}\left[(s(\eta)-\xi)^2\varphi\left(\frac{(s(\eta)-\xi)^2}{4\eta}\right)+(s(\eta)+\xi)^2\varphi\left(\frac{(s(\eta)+\xi)^2}{4\eta}\right)\right] \\ &-\frac{f(\xi)s'(\eta)}{4(\pi\eta^3)^{1/2}}\left[(s(\eta)-\xi)\varphi\left(\frac{(s(\eta)-\xi)^2}{4\eta}\right)+(s(\eta)+\xi)\varphi\left(\frac{(s(\eta)+\xi)^2}{4\eta}\right)\right]\Bigg\}\,d\xi. \end{aligned} \tag{17}$$

It can be proved easily that $g(0+) = 0$ and $\eta^{1/2}g'(\eta) \to f'(a)/2\pi^{1/2}$ as $\eta \to 0$. To prove (16)

$$\begin{aligned} \int_0^t\int_0^\tau \frac{g'(\eta)}{(\tau-\eta)^{1/2}}\,d\eta\,d\tau &= \int_0^t\int_\eta^t \frac{g'(\eta)}{(\tau-\eta)^{1/2}}\,d\tau\,d\eta \\ &= \int_0^t 2g'(\eta)(t-\eta)^{1/2}\,d\eta = \int_0^t \frac{g(\eta)}{(t-\eta)^{1/2}}\,d\eta. \end{aligned} \tag{18}$$

The last step follows by partial integration. On differentiating both sides of (18) we get (16).

We show next that

$$\frac{\partial}{\partial t}\int_\tau^t \frac{\psi(\eta,\tau)}{[(\eta-\tau)(t-\eta)]^{1/2}}\,d\eta = \int_\tau^t \frac{\psi_\eta(\eta,\tau)}{t-\tau}\left(\frac{\eta-\tau}{t-\eta}\right)^{1/2}\,d\eta, \tag{19}$$

where ψ is given by (9). From (14) $a + Bt \le s(t) \le a + At$; suppose T is chosen so that $a + BT > 0$. Thus $s(t) \ge a + BT > 0$. The function $\psi(\eta, \tau)$ is continuous in $t \ge \eta \ge \tau \ge 0$, with $\psi(\tau, \tau) = (4\pi)^{-1/2}$ and the derivative ψ_η given by

$$\begin{aligned} \psi_\eta(\eta,\tau) = \frac{1}{2\pi^{1/2}}&\left[-\frac{(s(\eta)-s(\tau))s'(\eta)}{2(\eta-\tau)}+\frac{(s(\eta)-s(t))^2}{4(\eta-\tau)^2}\right]\varphi\left(\frac{(s(\eta)-s(\tau))^2}{4(\eta-\tau)}\right) \\ +\frac{1}{2\pi^{1/2}}&\left[-\frac{(s(\eta)+s(\tau))s'(\eta)}{2(\eta-\tau)}+\frac{(s(\eta)+s(\tau))^2}{4(\eta-\tau)^2}\right]\varphi\left(\frac{(s(\eta)+s(\tau))^2}{4(\eta-\tau)}\right). \end{aligned} \tag{20}$$

$\psi_\eta(\eta, \tau)$ is again continuous for $t \ge \eta \ge \tau \ge 0$; it can be proved easily that in this region there is a constant K such that $|\psi_\eta(\eta, \tau)| \le K$ for any choice of $s(t)$ in the class described by (13).

The proof of (19) proceeds as follows:

$$\begin{aligned} \int_\tau^t\int_\tau^\gamma \frac{\psi_\eta(\eta,\tau)}{\gamma-\tau}\left(\frac{\eta-\tau}{\gamma-\eta}\right)^{1/2}\,d\eta\,d\gamma &= \int_\tau^t\int_\eta^t \frac{\psi_\eta(\eta,\tau)}{\gamma-\tau}\left(\frac{\eta-\tau}{\gamma-\eta}\right)^{1/2}\,d\gamma\,d\eta \\ = \int_\tau^t 2\psi_\eta(\eta,\tau)\tan^{-1}\left(\frac{t-\eta}{\eta-\tau}\right)^{1/2}\,d\eta &= \pi^{1/2} + \int_\tau^t \frac{\psi(\eta,\tau)}{[(\eta-\tau)(t-\eta)]^{1/2}}\,d\eta. \end{aligned} \tag{21}$$

The last step follows by partial integration. On differentiating (21) with respect to t we get (19).

2. Continuity of $F(s(t), \mu(t))$ and $G(s(t), \mu(t))$. The first term of F, the integral

with respect to τ, behaves essentially as $\int_0^t (t - \tau)^{-1/2} d\tau = 2t^{1/2}$ and is therefore continuous. The continuity of the second term, the integral with respect to ξ, follows from standard theorems. The continuity of the first term of G, the double integral, may be ascertained by introducing the variables $\tau = t\xi$, $\eta = t\zeta$, while the second term of G, the single integral, behaves as the first term of F.

3. Preservation of properties (13b, c) under the mapping (15). The preservation of properties (13b, c) follows from the fact, proved below, that $F(s(t), \mu(t))$ and $G(s(t), \mu(t))$ converge uniformly to $f'(a)/2$ and $f'(a)$ as $t \to 0$. By uniform convergence we mean uniform with respect to the class of functions described by (13b, c). More exactly we may make the differences $|F(s(t), \mu(t)) - f'(a)/2|$ and $|G(s(t), \mu(t)) - f'(a)|$ arbitrarily small by choosing t sufficiently small, and this choice of t is independent of $\mu(t)$ and $s(t)$ in the class of functions described by properties (13b, c). It follows then, from (15), that $\mu^*(0) = f'(a)$ and $\alpha s^{*\prime}(0)/\beta + Q(0)/\beta = f'(a)$ so that (13b) is preserved under the mapping (15). It also follows that T may be selected independently of $\mu(t)$ and $s(t)$ so that $|s^{*\prime}(t) - s'(0)| \leq \zeta_1$ and $|\mu^*(t) - f'(a)| \leq \zeta_2$ for $0 \leq t \leq T$; thus property (13c) is preserved under the mapping (15).

We proceed now with the proof of the uniform convergence of F and G to $f'(a)/2$ and $f'(a)$. The absolute value of the first term of G does not exceed

$$\gamma \int_0^t \int_\tau^t \frac{1}{t - \tau} \left(\frac{\eta - \tau}{t - \eta} \right)^{1/2} d\eta \, d\tau < \gamma \int_0^t \int_\tau^t \frac{d\eta \, d\tau}{[(t - \tau)(t - \eta)]^{1/2}} = 2t\gamma,$$

where $\gamma = 2\pi^{-1/2} K(f'(a) + \zeta_2)$. Thus the first term of G tends to 0 uniformly as $t \to 0$. Next, consider the second term of G; referring to (17) this second term is the sum of six terms of which the first is

$$-\frac{1}{\pi} \int_0^t \frac{1}{[\eta(t - \eta)]^{1/2}} \left\{ \int_0^a \frac{-f(\xi)}{2\eta} \varphi\left(\frac{(s(\eta) - \xi)^2}{4\eta} \right) d\xi \right\} d\eta. \tag{22}$$

The expression within the braces does not exceed

$$\int_0^{a+B\eta} \frac{-f(\xi)}{2\eta} \varphi\left(\frac{(a + B\eta - \xi)^2}{4\eta} \right) d\xi + \int_{a+B\eta}^a \frac{-f(\xi)}{2\eta} \, d\xi. \tag{23}$$

Let $\epsilon = -B(\eta + \eta^{1/4})$, where η is small enough that $\epsilon < \delta$. Let $M_1(\eta)$ and $M_2(\eta)$ be, respectively, the supremum in $0 < \xi \leq a$ and the maximum in $a + B\eta \leq \xi \leq a$ of $-f(\xi)$. Then (23) does not exceed

$$\int_0^{a-\epsilon} \frac{M_1}{2\eta} \varphi\left(\frac{(a + B\eta - \xi)^2}{4\eta} \right) d\xi$$
$$+ \int_{a-\epsilon}^{a+B\eta} \frac{-f(a + B\eta) + f'(\xi_1)(a + B\eta - \xi)}{2\eta} \varphi\left(\frac{(a + B\eta - \xi)^2}{4\eta} \right) d\xi + \frac{M_2(\eta)(-B)}{2},$$

where $\xi \leq \xi_1 \leq a + B\eta$. Let $M_3(\eta)$ be the maximum of $f'(\xi)$ in $a - \epsilon \leq \xi \leq a$; then $M_3(\eta) \to f'(a)$ as $\eta \to 0$ and

$$\int_{a-\epsilon}^{a+B\eta} \frac{-f(a + B\eta) + f'(\xi_1)(a + B\eta - \xi)}{2\eta} \varphi\left(\frac{(a + B\eta - \xi)^2}{4\eta} \right) d\xi$$
$$\leq \frac{-f(a + B\eta)}{2\eta} \int_{-\infty}^{\infty} \varphi\left(\frac{(a + B\eta - \xi)^2}{4\eta} \right) d\xi + M_3(\eta)\left[1 - \varphi\left(\frac{B^2}{4\eta^{1/2}} \right) \right]$$
$$= \frac{-f(a + B\eta)}{2\eta} (4\pi\eta)^{1/2} + M_3(\eta)\left[1 - \varphi\left(\frac{B^2}{4\eta^{1/2}} \right) \right].$$

Since, as $\eta \to 0$,

$$\int_0^{a-\epsilon} \frac{1}{2\eta}\,\varphi\left(\frac{(a+B\eta-\xi)^2}{4\eta}\right) d\xi \to 0, \qquad M_2(\eta) \to 0,$$

$$\frac{-f(a+B\eta)}{2\eta} \to \frac{Bf'(a)}{2}, \qquad \varphi\left(\frac{B^2}{4\eta^{1/2}}\right) \to 0,$$

the expression within the braces of (22) does not exceed $\omega_1(\eta) + f'(a)$, where $\omega_1(\eta)$ can be made arbitrarily small by choosing η sufficiently small; the function $\omega_1(\eta)$ does not depend on $s(\eta)$. A similar argument holds in the other direction. Thus

$$\omega_2(\eta) + f'(a) \le \int_0^a \frac{-f(\xi)}{2\eta}\,\varphi\left(\frac{(s(\eta)-\xi)^2}{4\eta}\right) d\xi \le \omega_1(\eta) + f'(a), \tag{24}$$

where $\omega_2(\eta)$ does not depend on $s(\eta)$ and can be made arbitrarily small by choosing η sufficiently small. From (24) it follows that (23) lies between the two functions of t

$$-\frac{1}{\pi}\int_0^t \frac{\omega_k(\eta)}{[\eta(t-\eta)]^{1/2}}\,d\eta - f'(a), \qquad k = 1, 2. \tag{25}$$

It follows from (25) that (22) converges uniformly to $-f'(a)$. A similar discussion shows that the second term (of the six making up the second term of G) converges to $2f'(a)$ uniformly; and finally the remaining four terms can be shown to go to 0 uniformly. Thus we may conclude that $G(s(t), \mu(t))$ tends to $f'(a)$ uniformly as $t \to 0$.

Considering now F, it is easily proved that the first integral of F tends uniformly to 0 as $t \to 0$. The second integral has two terms; the term involving $s(t) + \xi$ is also easily proved to go to 0 uniformly as $t \to 0$. There remains the term

$$\int_0^a \frac{-f(\xi)}{4(\pi t^3)^{1/2}}\,(s(t)-\xi)\varphi\left(\frac{(s(t)-\xi)^2}{4t}\right) d\xi. \tag{26}$$

If we decompose the interval as in (23), then it can be seen that (26) does not exceed a sum of terms which go to 0 uniformly with t plus the term

$$\begin{aligned}\int_{a-\epsilon}^{a+Bt} &\frac{f'(\xi_1)(a+Bt-\xi)}{(4\pi t)^{1/2}(2t)}\,(a+At-\xi)\varphi\left(\frac{(a+Bt-\xi)^2}{4t}\right) d\xi \\ &\le M_3(t)\left[(A-B)\left(\frac{t}{4\pi}\right)^{1/2} - \frac{(A-B)t - Bt^{1/4}}{(4\pi t)^{1/2}}\,\varphi\left(\frac{B^2}{4t^{1/2}}\right)\right. \\ &\qquad \left. + \int_{a-\epsilon}^{a+Bt} \frac{1}{(4\pi t)^{1/2}}\,\varphi\left(\frac{(a+Bt-\xi)^2}{4t}\right) d\xi\right].\end{aligned} \tag{27}$$

The integral on the right of (27) is equal to

$$\int_0^{-B/2t^{1/4}} (2\pi)^{-1/2}\varphi(\xi^2/2)\,d\xi,$$

and thus goes to $\frac{1}{2}$ as $t \to 0$. Since $M_3(t) \to f'(a)$ the expression (26) does not exceed a quantity which goes to $f'(a)/2$ uniformly as $t \to 0$. A similar argument can be given for the other direction; thus (26), and therefore also F, goes to $f'(a)/2$ uniformly as $t \to 0$.

4. Contraction property of the mapping (15). We consider first G, and consider the first term of the six which make up the second integral of G, i.e., the integral involving $g'(\eta)$. It must be shown that the absolute value of

$$\int_0^t \int_0^a \frac{-f(\xi)}{4[\pi\eta^3(t-\eta)]^{1/2}} \left[\varphi\left(\frac{(s_2(\eta)-\xi)^2}{4\eta}\right) - \varphi\left(\frac{(s_1(\eta)-\xi)^2}{4\eta}\right)\right] d\xi\, d\eta \tag{28}$$

does not exceed $c_1(T)||s_2 - s_1||$, where T may be chosen so that $c_1(T)$ is arbitrarily small. By the mean value theorem, (28) be written as

$$\int_0^t \int_0^a \frac{-f(\xi)(s_2(\eta) - s_1(\eta))}{4[\pi\eta^3(t-\eta)]^{1/2}} \left(-\frac{(s(\eta)-\xi)}{2\eta}\right)\varphi\left(\frac{(s(\eta)-\xi)^2}{4\eta}\right) d\xi\, d\eta, \tag{29}$$

where $s(\eta)$ lies between $s_1(\eta)$ and $s_2(\eta)$. Let $I(\xi, \eta)$ be the integrand in (29). From

$$s_2(t) - s_1(t) = \int_0^t (s_2'(\eta) - s_1'(\eta))\, d\eta$$

it follows that $|s_2(t) - s_1(t)| \leq T||s_2 - s_1||$ for $0 \leq t \leq T$, and, using the notation of the preceding section,

$$\begin{aligned} \left|\int_0^t \int_0^{a-\epsilon} I(\xi, \eta)\, d\xi\, d\eta\right| &\leq T\, ||s_2 - s_1|| \int_0^t \frac{M_1 a\eta^{-2}}{[16\pi\eta(t-\eta)]^{1/2}}\, \varphi\left(\frac{B^2}{4}\, \eta^{-1/4}\right) d\eta \\ &= T\, ||s_2 - s_1|| \int_0^1 \frac{M_1 a(t\zeta)^{-2}}{[16\pi\zeta(1-\zeta)]^{1/2}}\, \varphi\left(\frac{B^2}{4}\, (t\zeta)^{-1/4}\right) d\zeta. \end{aligned} \tag{30}$$

The function $x^{-2}\varphi(B^2x^{-1/4}/4)$ reaches its maximum at $x = (32/B^2)^4$; it is increasing in $0 \leq x \leq (32/B^2)^4$. Thus if $0 \leq t \leq T \leq (32/B^2)$ and $0 \leq \zeta \leq 1$,

$$(t\zeta)^{-2}\varphi\left(\frac{B^2}{4}\, (t\zeta)^{-1/4}\right) \leq T^{-2}\varphi\left(\frac{B^2}{4}\, T^{-1/4}\right),$$

so that the right side of (30) does not exceed $T^{-1}\varphi(B^2T^{-1/4}/4)(M_1 a\pi^{1/2}/4)||s_2 - s_1||$; the coefficient of $||s_2 - s_1||$ in this expression may be made arbitrarily small by taking T sufficiently small. We have also

$$\begin{aligned} \left|\int_0^t \int_{a-\epsilon}^a I(\xi, \eta)\, d\xi\, d\eta\right| &= \left|\int_0^t \frac{-f(a-\epsilon)(s_2(\eta) - s_1(\eta))}{4[\pi\eta^3(t-\eta)]^{1/2}}\, \varphi\left(\frac{(s(\eta) - a + \epsilon)^2}{4\eta}\right) d\eta \right. \\ &\qquad \left. - \int_0^t \int_{a-\epsilon}^a \frac{f'(\xi)(s_2(\eta) - s_1(\eta))}{4[\pi\eta^3(t-\eta)]^{1/2}}\, \varphi\left(\frac{(s(\eta)-\xi)^2}{4\eta}\right) d\xi\, d\eta\right|. \end{aligned} \tag{31}$$

By the mean value theorem $(s_2(\eta) - s_1(\eta))/\eta = s_2'(\eta^*) - s_1'(\eta^*)$, where η^* lies between 0 and η, so that $|(s_2(\eta) - s_1(\eta))/\eta| \leq ||s_2 - s_1||$ for $0 \leq \eta \leq t \leq T$. If $M_1(t)$ and $M_2(t)$ are the maxima of $|f(\xi)|$ and $|f'(\xi)|$ in $a + Bt + Bt^{1/4} \leq \xi \leq a$ then the right side of (31) does not exceed

$$\begin{aligned} &[M_1(t)\pi^{1/2}/4 + M_2(t)\, |B|\, (t + t^{1/4})\pi^{1/2}/4]\, ||s_2 - s_1|| \\ &\qquad \leq (\pi^{1/2}/4)[M_1(T) + M_2(T)\, |B|\, (T + T^{1/4})]\, ||s_2 - s_1||. \end{aligned} \tag{32}$$

The coefficient of $||s_2 - s_1||$ on the right of (32) may be made arbitrarily small by taking T sufficiently small. Thus the absolute value of (28) may be made less than $c_1(T)||s_1 - s_1||$, where $c_1(T)$ may be made arbitrarily small by appropriate choice of T. By similar arguments we can come to the same conclusion about the remaining five terms of the second integral of G.

Considering now the first integral of G we want to show that the absolute value of

$$\int_0^t \int_\tau^t \frac{1}{t-\tau}\left(\frac{\eta-\tau}{t-\eta}\right)^{1/2} (\psi_\eta^{(2)}(\eta, \tau)\mu_2(\tau) - \psi_\eta^{(1)}(\eta, \tau)\mu_1(\tau))\, d\eta\, d\tau \tag{33}$$

does not exceed $c_2(T)||s_2 - s_1|| + c_3(T)||\mu_2 - \mu_1||$, where $c_2(T)$ and $c_3(T)$ are positive and may be chosen arbitrarily small by taking T sufficiently small. Here $\psi_\eta^{(2)}$ and $\psi_\eta^{(1)}$ are obtained from (20) by replacing $s(\eta)$ and $s(\tau)$ by $s_2(\eta)$ and $s_2(\tau)$ and by $s_1(\eta)$ and $s_1(\tau)$. We may write (33) as

$$\int_0^t \int_\tau^t \frac{1}{t-\tau}\left(\frac{n-\tau}{t-\eta}\right)^{1/2} \psi_\eta^{(2)}(\mu_2 - \mu_1)\, d\eta\, d\tau$$
$$+ \int_0^t \int_\tau^t \frac{1}{t-\tau}\left(\frac{\eta-\tau}{t-\eta}\right)^{1/2} \mu_1(\psi_\eta^{(2)} - \psi_\eta^{(1)})\, d\eta\, d\tau. \tag{34}$$

The absolute value of the first term of (34) does not exceed

$$K\, ||\mu_2 - \mu_1|| \int_0^t \int_\tau^t \frac{d\eta\, d\tau}{[(t-\tau)(t-\eta)]^{1/2}} = 2tK\, ||\mu_2 - \mu_1|| \le 2TK\, ||\mu_2 - \mu_1||. \tag{35}$$

The coefficient of $||\mu_2 - \mu_1||$ on the right side of (35) can be made arbitrarily small by taking T sufficiently small. Introducing

$$\chi_k^{\pm}(\eta, \tau) = \varphi\left(\frac{(s_k(\eta) \pm s_k(\tau))^2}{4(\eta - \tau)}\right) \frac{1}{t-\tau}\left(\frac{\eta-\tau}{t-\eta}\right)^{1/2} \mu_1(\tau), \qquad k = 1, 2,$$

and referring to (20), the second term of (34) is the sum of four terms of which the first is (we omit the factor $-1/2\pi^{1/2}$)

$$\int_0^t \int_\tau^t \left[\frac{(s_2(\eta) - s_2(\tau))s_2'(\eta)}{2(\eta-\tau)} \chi_2^-(\eta, \tau) - \frac{(s_1(\eta) - s_1(\tau))s_1'(\eta)}{2(\eta - \tau)} \chi_1^-(\eta, \tau)\right] d\eta\, d\tau. \tag{36}$$

The expression (36) is the sum of the following three terms

$$\text{(a)} \qquad \int_0^t \int_\tau^t \frac{s_2(\eta) - s_2(\tau)}{2(\eta - \tau)} (s_2'(\eta) - s_1'(\eta))\chi_2^-(\eta, \tau)\, d\eta\, d\tau,$$

$$\text{(b)} \qquad \int_0^t \int_\tau^t \left(\frac{s_2(\eta) - s_2(\tau)}{2(\eta-\tau)} - \frac{s_1(\eta) - s_1(\tau)}{2(\eta - \tau)}\right) s_1'(\eta)\chi_2^-(\eta, \tau)\, d\eta\, d\tau, \tag{37}$$

$$\text{(c)} \qquad \int_0^t \int_\tau^t \frac{s_1(\eta) - s_1(\tau)}{2(\eta-\tau)} s_1'(\eta)(\chi_2^-(\eta, \tau) - \chi_1^-(\eta, \tau))\, d\eta\, d\tau.$$

By the mean value theorem,

$$\frac{s_k(\eta) - s_k(\tau)}{\eta - \tau} = s_k'(\zeta), \qquad \frac{s_2(\eta) - s_2(\tau)}{\eta - \tau} - \frac{s_1(\eta) - s_1(\tau)}{\eta - \tau} = s_2'(\zeta) - s_1'(\zeta)$$

for some ζ (not the same in each equation) between η and τ. We use also the fact that $|\varphi(z_1) - \varphi(z_2)| \le |z_1 - z_2|$ for z_1 and z_2 non-negative. Then the absolute value of (37a) and (37b) each does not exceed

$$\gamma\, |B|\, ||s_2 - s_1|| \int_0^t \int_\tau^t \frac{d\eta\, d\tau}{[(t-\tau)(t-\eta)]^{1/2}}, \tag{38}$$

where $\gamma = (f'(a) + \zeta_2)/2$. The integral in (38) is $2t \le 2T$. The absolute value of (37c) does not exceed

$$\left(\frac{\gamma B^2}{4}\right) \int_0^t \int_\tau^t \left| \frac{s_2(\eta) - s_2(\tau)}{\eta - \tau} - \frac{s_1(\eta) - s_1(\tau)}{\eta - \tau} \right|$$

$$\cdot |s_2(\eta) - s_2(\tau) + s_1(\eta) - s_1(\tau)| \frac{d\eta\, d\tau}{[(t - \tau)(t - \eta)]^{1/2}} \leq 2T\gamma a B^2 \, ||s_2 - s_1||. \tag{39}$$

The coefficients of $||s_2 - s_1||$ in (38) and the right side of (39) can be made arbitrarily small by taking T sufficiently small. Thus (36) can be made less than $||s_2 - s_1||$ multiplied by a function of T which can be made arbitrarily small by making T sufficiently small. The second of four terms which make up the second term of (34) can be treated in the same manner. The third term is

$$\int_0^t \int_\tau^t \left[\frac{(s_2(\eta) + s_2(\tau))s_2'(\eta)}{2(\eta - \tau)} \chi_2^+(\eta, \tau) - \frac{(s_1(\eta) + s_1(\tau))s_1'(\eta)}{2(\eta - \tau)} \chi_1^+(\eta, \tau) \right] d\eta\, d\tau. \tag{40}$$

In a manner analogous to (37a, b, c) the expression (40) is the sum of three terms, each of which is obtained from the corresponding term of (37) by replacing χ_k^- by χ_k^+ and $s_k(\eta) - s_k(\tau)$ by $s_k(\eta) + s_k(\tau)$. Let (37 a+, b+, c+) be these three terms. Then the absolute value of (37a+) does not exceed $af(t)||s_2 - s_1||$, where

$$f(t) = 2\gamma \int_0^t \int_\tau^t \frac{1}{\eta - \tau} \varphi\left(\frac{(a + BT)^2}{\eta - \tau}\right) \frac{1}{t - \tau} \left(\frac{\eta - \tau}{t - \eta}\right)^{1/2} d\eta\, d\tau,$$

and, since $|s_2(t) - s_1(t)| \leq T||s_2 - s_1||$, the absolute value of (37b+) does not exceed $2\,|B|\, Tf(t)\, ||s_2 - s_1||$. Introducing the variables $\tau = t\xi$, $\eta = t\zeta$ we see that $f(t)$ is continuous, increasing, and $f(0) = 0$. Thus the absolute values of (37a+) and (37b+) do not exceed $af(T)\, ||s_3 - s_1||$ and $2\,|B|\, Tf(T)\, ||s_2 - s_1||$ and both coefficients of $||s_2 - s_1||$ go to 0 with T. We may write (37c+), using the mean value theorem and $\varphi'(\xi) = -\varphi(\xi)$,

$$\int_0^t \int_\tau^t s_1'(\eta) \frac{s_1(\eta) + s_1(\tau)}{2(\eta - \tau)} (-\varphi(\xi)) \left[\frac{(s_2(\eta) + s_2(\tau))^2}{4(\eta - \tau)} - \frac{(s_1(\eta) + s_1(\tau))^2}{4(\eta - \tau)} \right] \frac{1}{t - \tau} \left(\frac{\eta - \tau}{t - \eta}\right)^{1/2} \mu_1(\tau)\, d\eta\, d\tau,$$

where ξ lies between $(s_2(\eta) + s_2(\tau))^2/4(\eta - \tau)$ and $(s_1(\eta) + s_1(\tau))^2/4(\eta - \tau)$. Since either of these is greater than or equal to $(a + BT)^2/(\eta - \tau)$ we have

$$\varphi(\xi) \leq \varphi((a + BT)^2/(\eta - \tau)).$$

Thus the absolute value of (37c+) does not exceed

$$4\gamma\, |B|\, T\, ||s_2 - s_1|| \int_0^t \int_\tau^t \frac{a^2}{(\eta - \tau)^2} \varphi\left(\frac{(a + BT)^2}{\eta - \tau}\right) \frac{1}{t - \tau} \left(\frac{\eta - \tau}{t - \eta}\right)^{1/2} d\eta\, d\tau. \tag{41}$$

Introducing the variables ξ, ζ into the integral in (41) we see that, as a function of t, the integral is continuous and increasing in the vicinity of $t = 0$ and equal to 0 for $t = 0$. Thus we may make the coefficient of $||s_2 - s_1||$ in (41) as small as we wish by taking T sufficiently small. Thus it follows that (40) can be made less than $||s_2 - s_1||$ multiplied by a function of T which can be made arbitrarily small by making T sufficiently small. The fourth term of the four which made up the second term of (34) can be treated similarly. Thus our assertion about (33) is proved; it follows therefore that

$$||\mu_2^* - \mu_1^*|| \leq c(T)(||s_2 - s_1|| + ||\mu_2 - \mu_1||), \tag{42}$$

where T may be chosen so that $c(T)$ is arbitrarily small. By similar arguments on F we may establish that

$$||s_2^* - s_1^*|| \leq d(T)(||s_2 - s_1|| + ||\mu_2 - \mu_1||). \tag{43}$$

By adding (42) and (43) and by choosing T so that $c(T) + d(T)$ is positive and less than 1 we establish the contracting property of the mapping (15) in the interval $0 \leq t \leq T$.

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