BOUNDS ON λ FOR POSITIVE SOLUTIONS OF
$\Delta\psi + \lambda f(\mathbf{r})\{\psi + G(\psi)\} = 0.$*

BY

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We shall show that when $G(\psi) \geq G(0) = 1$ and ψ satisfies typical conditions on the closed (sufficiently smooth) boundary S of an open n dimensional region $\mathcal{V}$, the values of $\lambda > 0$ for which the title equation has positive solutions, ψ, are bounded above by the number

$$\eta_0 \max_{\varphi \geq 0} \frac{\varphi}{\varphi + G(\varphi)}$$

and by the function of $\psi_M = \max \psi$

$$\eta_0 \psi_M / (\psi_M + 1).$$

Thus, positive solutions of the title problem can exist only for values of $\lambda > 0$ satisfying the composite inequality (see Fig. 1)
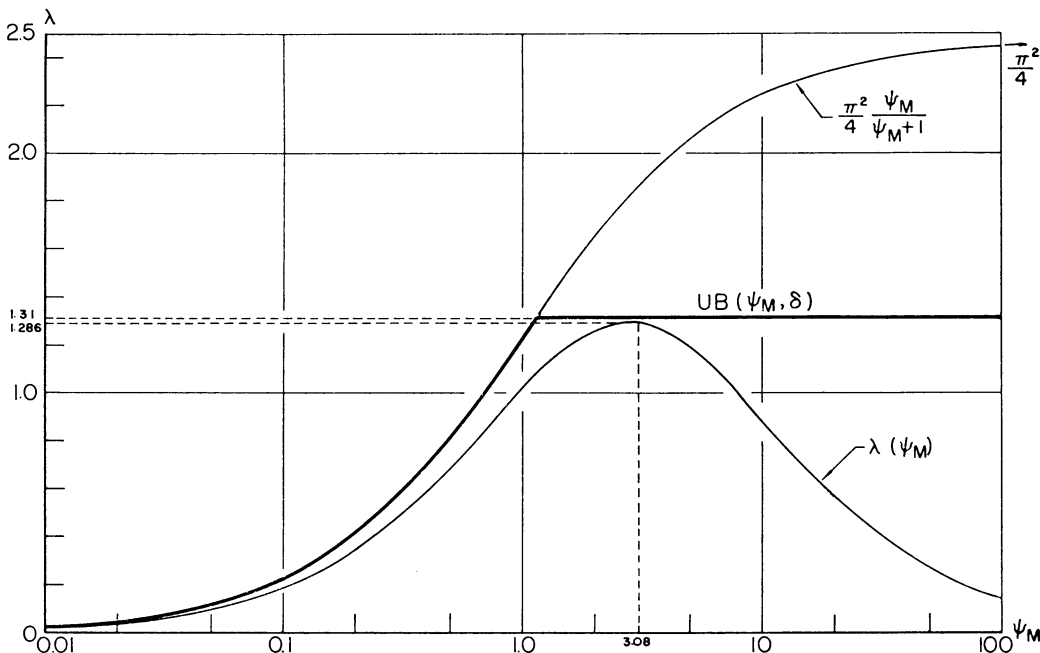

FIG. 1. Comparison of Exact Solution for $\lambda(\psi_M)$ with the Bound (1) for $\psi + G(\psi) = 1 + \psi + \delta\psi^2$, $\delta = .195$.

$$\lambda(\psi_M) \le UB(\psi_M) = \eta_0 \min \begin{cases} \max_{\varphi \ge 0} \varphi/(\varphi + G(\varphi)) \\ \psi_M/(\psi_M + 1). \end{cases} \tag{1}$$

Here $\mathbf{r}$ is the n dimensional position vector, $0 \not\equiv f(\mathbf{r}) \ge 0$ in $\mathcal{V}$, and $\eta_0 > 0$ and ψ'_0 are the least eigenvalue and positive eigenfunction, respectively, of the homogeneous equation

$$\Delta\psi' + \eta f(\mathbf{r})\psi' = 0 \tag{2}$$

corresponding to the title equation and suitably defined boundary conditions.

The one-dimensional version of the equation

$$\Delta\psi + \lambda f(\mathbf{r})(\psi + G(\psi)) = 0 \tag{3}$$

arises in applications involving the diffusion of heat generated by positive temperature-dependent sources. For example, the generation is given by Joule losses in electrically conducting solids, with λ representing the square of the (constant) current and $\psi + G(\psi)$ the temperature-dependent resistance [1], or by frictional heating, with λ representing the square of the (constant) shear stress and $\psi + G(\psi)$ the temperature-dependent fluidity [2]. The nonlinear equations for some of these simple one-dimensional cases can be integrated explicitly. Positive solutions exist for all values of $0 \le \psi \le \max \psi = \psi_M < \infty$ but only for a bounded set of λ. For these values of λ the solutions are double-valued. For these problems the bounds (1) are quite sharp. We compare the limiting λ as given by direct integration and by (1) in the concluding paragraphs.

We now turn to examine the conditions under which (1) is valid. A prescribed boundary condition of the third kind is considered first. The result here is:

THEOREM 1. Let $\mathcal{V}$ be a bounded and open n dimensional region and $\mathcal{S}$ its (sufficiently smooth) boundary. Assume that $\psi \epsilon C^2(\mathcal{V})$. Let equation (3), where Δ is an n dimensional Laplace operator and $0 \le f(\mathbf{r})\epsilon C(\mathcal{V})$, $f \not\equiv 0$ govern the behavior of ψ in $\mathcal{V}$, and

$$\alpha \frac{\partial\psi}{\partial n} + \beta\psi = 0 \quad \text{on} \quad \mathcal{S}, \tag{4}$$

where α, β, $\epsilon C^0_p(\mathcal{S})$ and the differentiation is along the outward normal to $\mathcal{S}$. Further, let $\psi \ge 0$ and

$$G(\psi) \ge G(0) = 1. \tag{5}$$

Then, if the homogeneous linear equation

$$\Delta\psi' + \eta f(\mathbf{r})\psi' = 0 \tag{6}$$

and the boundary condition (4) generate a positive eigen-function ψ'_0 and a least eigenvalue $\eta_0 > 0$, positive ψ cannot exist when $\lambda(\psi_M)$ is greater than the smallest of

$$\left\{ \frac{\eta_0\psi_M}{\psi_M + 1} , \eta_0 \max_{\varphi \ge 0} \frac{\varphi}{\varphi + G(\varphi)} \right\},$$

where $0 \le \psi \le \max \psi = \psi_M$.

Proof of Theorem 1: For any positive solution of (3) and (4) we have

$$0 = \int_{\mathcal{V}} (\psi'_0\Delta\psi - \psi\Delta\psi'_0) = (\eta_0 - \lambda) \int_{\mathcal{V}} f\psi'_0\psi - \lambda \int_{\mathcal{V}} f\psi'_0 G(\psi). \tag{7}$$

By equations (5) and (6)

$$\frac{\lambda}{\eta_0} = \begin{cases} \dfrac{\displaystyle\int_{\upsilon} f\psi_0'\psi \dfrac{\psi + G(\psi)}{\psi + G(\psi)}}{\displaystyle\int_{\upsilon} f\psi_0'(\psi + G(\psi))} \leq \max_{\varphi \geq 0} \left(\dfrac{\varphi}{\varphi + G(\varphi)} \right) \\[4ex] 1 \Big/ \left[1 + \dfrac{\displaystyle\int_{\upsilon} f\psi_0'G(\psi)}{\displaystyle\int_{\upsilon} f\psi_0'\psi} \right] \leq \dfrac{1}{1 + 1/\psi_M} = \dfrac{\psi_M}{\psi_M + 1}. \end{cases} \tag{8}$$

The case

$$\psi = 0 \quad \text{on} \quad \mathcal{S} \tag{9}$$

occurs frequently in the applications. For this case there exists $\eta_0 > 0$.

Two extensions of Theorem 1 may be easily obtained:

THEOREM 2. Let the self-adjoint operator L (of any order) and boundary conditions replace Δ and (4), respectively, in Theorem 1. The modified theorem is valid.

THEOREM 3. Let the condition

$$\psi = \psi(\mathcal{S}) \quad \text{prescribed on} \quad \mathcal{S} \tag{10}$$

replace (4) in Theorem 1. Let Min $\psi = 0$ be a value of $\psi(\mathcal{S})$. Let η_0 and ψ_0' be defined by (2) and (9). The modified theorem is valid.

Proof of Theorem 3. Introduce a harmonic function φ such that

$$\varphi(\mathcal{S}) = \psi(\mathcal{S}) \tag{11}$$

and

$$\Gamma = \psi - \varphi \tag{12}$$

Of course φ is not negative in $\upsilon + \mathcal{S}$ and Min $\varphi = $ Min $\psi = 0$. Moreover, if ψ is positive, it follows from equation (5) that ψ is superharmonic [3] in υ; i.e.,

$$\psi \geq \varphi \quad \text{in} \quad \upsilon. \tag{13}$$

Hence, if $\psi \geq 0$ then $\Gamma \geq 0$. Conversely, a positive Γ implies a positive ψ.

Equations (3) and (12) may be combined to produce

$$\Delta\Gamma + \lambda(\Gamma + F(\varphi, \Gamma)) = 0, \tag{14}$$

$$\Gamma(\mathcal{S}) = 0, \tag{15}$$

$$F(\varphi, \Gamma) = \varphi + G(\Gamma + \varphi) \geq G(0) = 1. \tag{16}$$

This is a variant of the problem treated in Theorem 1.

Inequality (1) is the principal result of this investigation. The simple form of this inequality is not deceptive. All of the complications introduced by various boundary conditions, the dimensionality of υ and the functional form of $f(\mathbf{r})$ are absorbed in the determination of the parameter η_0 from the eigenvalue problem associated with the *linear* equation (2) and natural boundary conditions. The results of course depend critically on the existence of a least positive eigenvalue (and positive eigenfunction) for the linear problem. This does not seriously restrict the scope of the result.

We conclude with an examination of the implications of (1) in some simple cases.

a. With $G(\psi) \equiv 1$

$$\lambda(\psi_M) \leq \frac{\eta_0 \psi_M}{\psi_M + 1},$$

and $\lambda(\psi_M)$ is monotone with a maximum $\lambda = \eta_0$ as $\psi_M \to \infty$. The first eigenvalue of the homogeneous linear system is an upper bound on the linear non-homogeneous system.

b. The problem

$$\frac{d^2\psi}{dx^2} + \lambda e^\psi = 0$$

$$\psi(\pm 1) = 0$$

has a double-valued solution [2] for

$$\lambda < \max \lambda(\psi_M) = \lambda_{\max} = .893$$

and no solutions when λ is greater than $\lambda_{\max}$. The bound (1) gives

$$\lambda_{\max} \leq \max UB(\psi_M) = \pi^2/4e = .91.$$

c. The problem

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{d\psi}{dr}\right) + \lambda r^2 e^\psi = 0$$

$$\psi(1) = \frac{d\psi}{dr}(0) = 0$$

has a double-valued solution [4] when

$$\lambda < \lambda_{\max} = 8.0$$

and no solution when $\lambda > \lambda_{\max}$. The bound (1) gives

$$\lambda_{\max} \leq \max UB(\psi_M) = (4.81)^2/e = 8.5$$

where $(4.81)^2 = \eta_0$ is determined by the first positive root of $J_0(\eta_0^{\frac{1}{2}}/2) = 0$.

d. The problem

$$\frac{d^2\psi}{dx^2} + \lambda(1 + \psi + \delta\psi^2) = 0$$

$$\psi(\pm 1) = 0$$

has a double-valued solution [2] for every non-negative δ when

$$\lambda < \lambda_{\max}(\delta) = \max_{\psi_M \geq 0}\left\{\frac{3\cos\gamma}{2\delta\sigma}\left(\int_0^B \frac{d\Phi}{(1 - K^2\sin^2\Phi)^{\frac{1}{2}}}\right)^2\right\}, \tag{17}$$

where

$$\sigma^2 = \tfrac{3}{4}(\psi_M^2 + \psi_M/\delta + 4/\delta - \tfrac{3}{4}\delta^2),$$

$$\text{Tan } \gamma = 3(\psi_M + \tfrac{1}{2}\delta)/2\sigma,$$

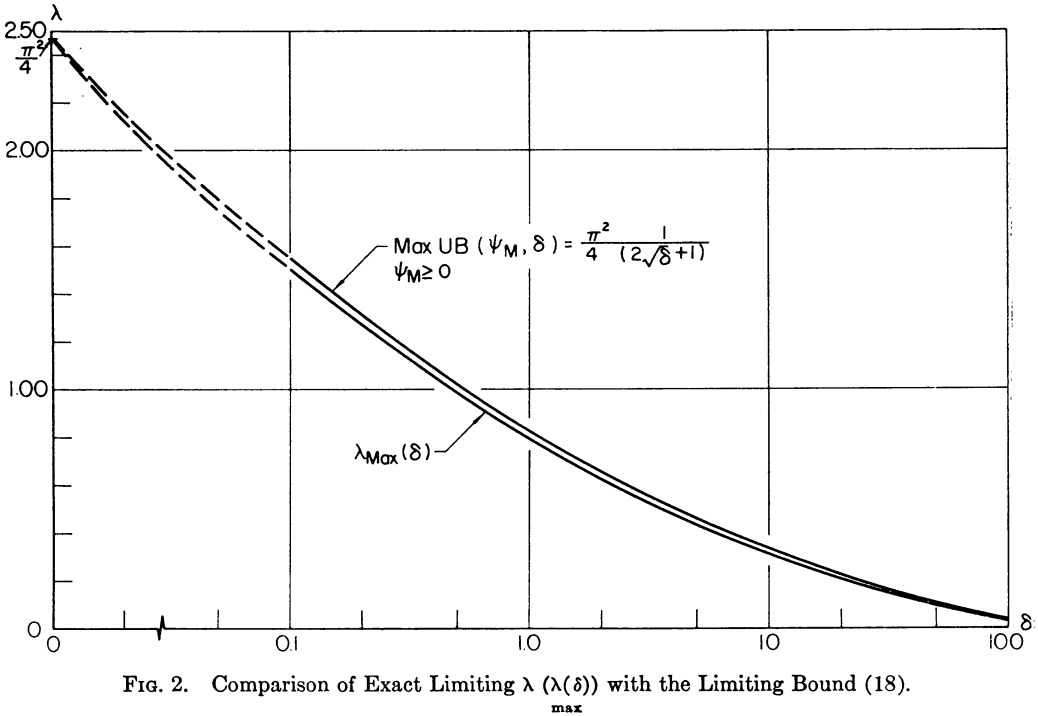$$K^2 = \tfrac{1}{2}(1 + \sin\gamma),$$

and

FIG. 2.   Comparison of Exact Limiting λ ($\lambda(\delta)$) with the Limiting Bound (18).

$$B = 2 \operatorname{Tan}^{-1} [\psi_M \cos \gamma / \sigma]^{\frac{1}{2}}.$$

The bound (1) gives

$$\lambda_{\max} (\delta) \leq \max_{\psi_M \geq 0} UB(\psi_M , \delta) = \frac{\pi^2}{4(2\sqrt{\delta} + 1)}. \tag{18}$$

In Fig. 1 we have compared the bound $UB(\psi_M, \delta)$ for $\delta = .195$ with $\lambda(\psi_M)$ as calculated from the exact solution. In Fig. 2 we have compared the exact limiting values $\lambda_{\max}(\delta)$ as calculated from (17) with the dominating values $\pi^2/4(2\sqrt{\delta} + 1)$.

In these simple examples solutions exist for $\psi_M < \infty$ but for a bounded set of λ. A sufficiently small λ is equivalent to a sufficiently small domain with a suitably rescaled metric. It follows from existence theorems of Courant and Hilbert [5] that solutions to the title problem for bounded values of $\psi(1 \leq \psi + G(\psi) \leq \psi_M + G(\psi_M))$ will exist for all $\psi_M < \infty$ and condition (10).

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