TRANSVERSE ELECTRIC AND MAGNETIC EFFECTS*

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Abstract. The non-linear electromagnetic constitutive equations, for an isotropic material possessing a center of symmetry, are used to predict the possibility of certain new effects.

1. Introduction. In this paper, the non-linear electromagnetic constitutive equations, for an isotropic material possessing a center of symmetry, previously developed [1] are used to predict the possibility of certain effects which, as far as we are aware, have not so far been observed. Thus, if we assume that the electric field is a function of the electric displacement and magnetic induction vectors, an electric field may result perpendicular to the plane of these two vectors. Again, if the magnetic intensity field is a function of the electric displacement and magnetic induction vectors, a magnetic field may develop at right angles to these two vectors. In the case when the material is conservative, i.e., possesses an electromagnetic energy density function which depends on the electric displacement and magnetic induction fields only, both of these effects are zero.

2. The transverse effects. In a previous paper [1], it has been shown that if we assume that the current density vector $\mathbf{J}$ and the magnetic intensity vector $\mathbf{H}$ in an isotropic material possessing a center of symmetry are functions of the electric vector $\mathbf{E}$ and magnetic induction vector $\mathbf{B}$, then**

$$\mathbf{J} = \gamma_1'\mathbf{E} + \gamma_2'\mathbf{E} \times \mathbf{B} + \gamma_3'(\mathbf{E}\cdot\mathbf{B})\mathbf{B}$$

and

$$\mathbf{H} = \beta_1'\mathbf{B} + \beta_2'(\mathbf{E}\cdot\mathbf{B})\mathbf{E} + \beta_3'(\mathbf{E}\cdot\mathbf{B})\mathbf{E} \times \mathbf{B}, \tag{2.1}$$

where the γ''s and β''s are functions of $\mathbf{E}\cdot\mathbf{E}$, $\mathbf{B}\cdot\mathbf{B}$, $(\mathbf{E}\cdot\mathbf{B})^2$. If we assume that the electric displacement vector $\mathbf{D}$ is a function of $\mathbf{E}$ and $\mathbf{B}$, then it follows by reasoning identical with that leading to the first of equations (2.1) that

$$\mathbf{D} = \alpha_1'\mathbf{E} + \alpha_2'\mathbf{E} \times \mathbf{B} + \alpha_3'(\mathbf{E}\cdot\mathbf{B})\mathbf{B}, \tag{2.2}$$

where the α''s are functions of $\mathbf{E}\cdot\mathbf{E}$, $\mathbf{B}\cdot\mathbf{B}$ and $(\mathbf{E}\cdot\mathbf{B})^2$.

An alternative form for the equations (2.1) and (2.2) may be obtained by a mathematically analogous argument if we assume that $\mathbf{E}$ and $\mathbf{H}$ are functions of $\mathbf{D}$ and $\mathbf{B}$, thus

$$\mathbf{E} = \mathbf{E}(\mathbf{D}, \mathbf{B}), \qquad \mathbf{H} = \mathbf{H}(\mathbf{D}, \mathbf{B}). \tag{2.3}$$

We then find that, for an isotropic material possessing a center of symmetry,

$$\begin{aligned} \mathbf{E} &= \alpha_1\mathbf{D} + \alpha_2\mathbf{D} \times \mathbf{B} + \alpha_3(\mathbf{D}\cdot\mathbf{B})\mathbf{B}, \\ \mathbf{H} &= \beta_1\mathbf{B} + \beta_2(\mathbf{D}\cdot\mathbf{B})\mathbf{D} + \beta_3(\mathbf{D}\cdot\mathbf{B})\mathbf{D} \times \mathbf{B}, \\ \mathbf{J} &= \gamma_1\mathbf{D} + \gamma_2\mathbf{D} \times \mathbf{B} + \gamma_3(\mathbf{D}\cdot\mathbf{B})\mathbf{B}, \end{aligned} \tag{2.4}$$

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**In the previous paper it was actually shown that if $\mathbf{J}$ and $\mathbf{H}$ are polynomial functions of $\mathbf{E}$ and $\mathbf{B}$, then the expressions (2.1), with the γ''s and β''s polynomial functions of $\mathbf{E}\cdot\mathbf{E}$, $\mathbf{B}\cdot\mathbf{B}$, $(\mathbf{E}\cdot\mathbf{B})^2$, are valid. The generalization stated here follows immediately by using a theorem due to Wineman and Pipkin[2].

where the α's, β's and γ's are functions of $\mathbf{D}\cdot\mathbf{D}$, $\mathbf{B}\cdot\mathbf{B}$ and $(\mathbf{B}\cdot\mathbf{D})^2$. Equations (2.4) may also be obtained by inversion of (2.1) and (2.2). If the dependence of **E** and **H** on **D** and **B** indicated in (2.3) is polynomial dependence, then the α's, β's and γ's in (2.4) are polynomials in $\mathbf{D}\cdot\mathbf{D}$, $\mathbf{B}\cdot\mathbf{D}$ and $(\mathbf{D}\cdot\mathbf{B})^2$.

If **D** and **B** are not parallel than it follows from the third of equations (2.4) that **J** has, in general, a component in the direction perpendicular to the plane of **D** and **B**. This is, of course, the well-known Hall effect and occurs even if in the constitutive equations we neglect terms of higher degree than the second in **D** and **B**.

The first of equations (2.4) indicates the possibility that if **B** and **D** are not parallel, **E** may have a component in the direction perpendicular to the plane of **B** and **D**. We are not aware that such an effect, which we may call the *transverse electric effect,* has been observed.

The second of equations (2.4) indicates the possibility that if **B** and **D** are neither parallel nor perpendicular, **H** may have a component in the direction perpendicular to the plane of **B** and **D**. Again, we are not aware that such an effect, which we may call the *transverse magnetic effect* has been observed.

Now, suppose that the α's, β's and γ's are polynomials in $\mathbf{D}\cdot\mathbf{D}$, $\mathbf{B}\cdot\mathbf{B}$ and $(\mathbf{B}\cdot\mathbf{D})^2$, and we assume that **B** and **D** are small. We may then obtain an Nth order approximation to each of the constitutive equations (2.4) by neglecting terms of degree greater than N in **B** and **D**. We note that second-order constitutive equations yield a transverse electric effect but not a transverse magnetic effect. The transverse magnetic effect is obtained only with approximations to the constitutive equation for **H** of order four or greater.

3. Conservative systems. We shall now assume that the material is ideally non-conducting and shall consider only isothermal behavior. Then, the energy balance equation for a region V of the material bounded by the surface A is

$$-\int_A (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{A} = \int_V (\mathbf{H}\cdot\dot{\mathbf{B}} + \mathbf{E}\cdot\dot{\mathbf{D}})\, dV, \tag{3.1}$$

where $d\mathbf{A}$ is a vector element of area of the surface A, and we use the dot to denote differentiation with respect to time. The term on the left-hand side of equation (3.1) represents the rate at which electromagnetic energy flows into the region V and the term on the right-hand side represents the rate of increase of electromagnetic energy in the region. Integrating (3.1) with respect to time over the interval t_1 to t_2, we obtain

$$-\int_{t_1}^{t_2} dt \int_A (\mathbf{E}\times\mathbf{H})\cdot d\mathbf{A} = \int_{t_1}^{t_2} dt \int_V (\mathbf{H}\cdot\dot{\mathbf{B}} + \mathbf{E}\cdot\dot{\mathbf{D}})\, dV. \tag{3.2}$$

The term on the left-hand side of (3.2) represents the energy flowing into the region V in the interval t_1 to t_2 and that on the right-hand side represents the increase in electromagnetic energy in the region in this interval.

Now suppose that **E** and **H** are each single-valued functions of **D** and **B**. We consider that **D** and **B** undergo closed cycles of variation in the time interval t_1 to t_2, so that their initial and final values are the same. The integral on the right-hand side of (3.2) must then be zero. If it were not, the total flow of electromagnetic energy into V would be either positive or negative. If it were positive, then by taking **D** and **B** round the closed cycle in the opposite direction, the flow of energy into V would become negative. Repeated execution of such a cycle would enable us to continually extract energy from the volume. Since this is not possible, we have

$$\oint \int_V (\mathbf{H}\cdot d\mathbf{B} + \mathbf{E}\cdot d\mathbf{D})\, dV = 0 \tag{3.3}$$

for all closed cycles of **B** and **D**. The relation (3.3) is, of course, valid for all regions V of the material and hence, since **E** and **H** are functions of **B** and **D** only, we may write $\mathbf{E}\cdot d\mathbf{D} + \mathbf{H}\cdot d\mathbf{B}$ as a total differential, thus:

$$dW(\mathbf{B}, \mathbf{D}) = \mathbf{H}\cdot d\mathbf{B} + \mathbf{E}\cdot d\mathbf{D}. \tag{3.4}$$

Whence, we obtain

$$\mathbf{E} = \partial W/\partial \mathbf{D}, \qquad \mathbf{H} = \partial W/\partial \mathbf{B}. \tag{3.5}$$

For an isotropic material possessing a center of symmetry, it is easily shown that W must depend on **B** and **D** through the isotropic invariants I_1, I_2, I_3 defined by

$$I_1 = \mathbf{D}\cdot\mathbf{D}, \qquad I_2 = \mathbf{B}\cdot\mathbf{B}, \qquad I_3 = (\mathbf{B}\cdot\mathbf{D})^2. \tag{3.6}$$

Then, from (3.5), we have

$$\begin{aligned} \mathbf{E} &= 2\left[\frac{\partial W}{\partial I_1}\mathbf{D} + \frac{\partial W}{\partial I_3}(\mathbf{B}\cdot\mathbf{D})\mathbf{B}\right], \\ \mathbf{H} &= 2\left[\frac{\partial W}{\partial I_2}\mathbf{B} + \frac{\partial W}{\partial I_3}(\mathbf{B}\cdot\mathbf{D})\mathbf{D}\right]. \end{aligned} \tag{3.7}$$

Comparing equations (3.7) and (2.4), we see that for a conservative system

$$\frac{\partial \alpha_1}{\partial I_3} = \frac{\partial \alpha_3}{\partial I_1}, \qquad \frac{\partial \alpha_1}{\partial I_2} = \frac{\partial \beta_1}{\partial I_1}, \qquad \frac{\partial \alpha_3}{\partial I_2} = \frac{\partial \beta_1}{\partial I_3}, \qquad \alpha_2 = \beta_3 = 0, \qquad \alpha_3 = \beta_2\,. \tag{3.8}$$

We note that in this case neither the transverse electric nor the transverse magnetic effect can exist.

4. Materials with memory. We now replace the constitutive assumption (2.3) by the assumption that the values of **E** and **H** at the instant of measurement t depend on $\mathbf{D}(\tau)$ and $\mathbf{B}(\tau)$, the values of the electric displacement and magnetic induction vector at time τ, for all times τ up to and including the instant of measurement; i.e., **E** and **H** are functionals of $\mathbf{D}(\tau)$ and $\mathbf{B}(\tau)$ thus:

$$\mathbf{E} = \mathbf{E}\underset{\tau=-\infty}{\overset{t}{[}}\mathbf{D}(\tau), \mathbf{B}(\tau)], \qquad \mathbf{H} = \mathbf{H}\underset{\tau=-\infty}{\overset{t}{[}}\mathbf{D}(\tau), \mathbf{B}(\tau)]. \tag{4.1}$$

We now restrict our considerations to electric displacement and magnetic induction fields which vary with time in a specified manner, thus:

$$\mathbf{D}(\tau) = \varphi(\tau)\mathbf{D}', \qquad \mathbf{B}(\tau) = \psi(\tau)\mathbf{B}', \tag{4.2}$$

where $\mathbf{D}'$ and $\mathbf{B}'$ are independent of time. Then, we may replace the functional dependence of **E** and **H** on $\mathbf{D}(\tau)$ and $\mathbf{B}(\tau)$ by function dependence on $\mathbf{D}'$ and $\mathbf{B}'$, this function dependence depending on the functions φ and ψ. If we now make the assumption that the material is isotropic and possesses a center of symmetry, the constitutive equations for **E** and **H** take the forms

$$\begin{aligned} \mathbf{E} &= \alpha_1\mathbf{D}' + \alpha_2\mathbf{D}'\times\mathbf{B}' + \alpha_3(\mathbf{D}'\cdot\mathbf{B}')\mathbf{B}', \\ \mathbf{H} &= \beta_1\mathbf{B}' + \beta_2(\mathbf{D}'\cdot\mathbf{B}')\mathbf{D}' + \beta_3(\mathbf{D}'\cdot\mathbf{B}')\mathbf{D}'\times\mathbf{B}', \end{aligned} \tag{4.3}$$

where the α's and β's are now functions of $\mathbf{D}' \cdot \mathbf{D}'$, $\mathbf{B}' \cdot \mathbf{B}'$, $(\mathbf{D}' \cdot \mathbf{B}')^2$ and t and depend on φ and ψ. These equations indicate the possibility of both transverse electric and magnetic effects.

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References

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2. A. S. Wineman and A. C. Pipkin, J. Rat'l Mech. Anal. **12**, 420 (1963)

Corrections to the paper

FINITE PURE BENDING OF CIRCULAR CYLINDRICAL TUBES

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The numerical values in Table 2 of this paper should be corrected in such a way that Table 2 now reads

	2 terms	3 terms	4 terms	numerical solution
α_c	1.633	1.439	1.541	1.66
m_c	1.089	1.002	1.034	1.06

The above values of α_c and m_c are in agreement with the corresponding values in Figure 2 of the original paper.