APPLICATION OF CONSERVATION LAWS TO THE ASYMPTOTIC PROPERTIES OF HYPERSONIC FLOW*

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Summary. The conservation laws involving mass, momentum, and energy are utilized for the derivation of a set of exact relations between the drag force on a body moving at hypersonic speeds through air and certain integral properties of the wake at great distances downstream. The above integral properties involve the surface integrals of the deviation of the hydrodynamical variables from their ambient values. The surface in question is a plane perpendicular to the wake axis. Explicitly, the relations state that all of the surface integrals just defined are proportional to the drag force with proportionality factors that are given functions of the ambient values of the hydrodynamical field variables. The results are valid if the plane cutting the wake is sufficiently far downstream that the deviations of the field variables are small relative to their ambient values everywhere on this plane and if irreversible processes are negligible there.

1. Introduction. This paper applies the conservation laws pertaining to mass, momentum, and energy to the determination of certain integral properties of the asymptotic part of the wake of a body moving at hypersonic speed. On a surface cutting across the wake is sufficiently far downstream, it can be assumed that the hydrodynamical field variables deviate little from their ambient values and that dissipative processes are occurring to a negligible extent. In this case a set of equations can be derived that express linear relations between the drag force and surface integrals of the deviations of the field variables from their ambient values. These relations, although simple, do not appear to be well known.

In the model a fictitious body force, equal and opposite to the drag force, keeps the body moving with constant velocity in a homogeneous atmosphere.

2. Integral form of the conservation laws. Let us consider a frame of reference with a fixed closed surface S moving with the body. Furthermore, let us assume that the body is located inside the closed surface but sufficiently far from it that on it the atmosphere is in local equilibrium and involves negligible transport effects (i.e., negligible shear stress, pressure given by equation of state, negligible heat flow, etc.). The conservation of mass, momentum, and energy is expressed by the following integral relations:

$$\int d\mathbf{S}\cdot\rho\mathbf{u} = 0 \tag{1}$$

$$\int d\mathbf{S}\cdot(\rho\mathbf{u}\mathbf{u} + \mathbf{I}p) = -\mathbf{F}_d \tag{2}$$

$$\int d\mathbf{S}\cdot\rho\mathbf{u}(h + \tfrac{1}{2}u^2) = 0 \tag{3}$$

where the integrations span the closed surface S. The symbols are defined as follows:

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$d\mathbf{S} = \mathbf{n}dS$, $\mathbf{n} \equiv$ outward pointing normal vector and dS is a scalar area element, $\rho \equiv$ density, $\mathbf{u} \equiv$ velocity (with respect to body), $p \equiv$ pressure, $h \equiv$ specific enthalpy (all of the quantities ρ, $\mathbf{u}$, p, and h pertain to air at an arbitrary position, either in the incident ambient stream or in the wake), $\mathbf{I} \equiv$ unit tensor, $\mathbf{F}_d \equiv$ drag force on body.

Using the relation

$$\int d\mathbf{S} = 0, \tag{4}$$

and remembering (1), we write

$$\int d\mathbf{S} \cdot \rho_0 \mathbf{u}_0 = 0, \tag{5}$$

$$\int d\mathbf{S} \cdot (\rho \mathbf{u}\mathbf{u}_0 + \mathbf{I}p_0) = 0, \tag{6}$$

$$\int d\mathbf{S} \cdot \rho \mathbf{u}(h_0 + \tfrac{1}{2}u_0^2) = 0, \tag{7}$$

where the subscript 0 denotes the properties of the ambient atmosphere. Subtracting (5), (6), and (7) from (1), (2), and (3), respectively, we obtain

$$\int d\mathbf{S} \cdot (\rho \mathbf{u} - \rho_0 \mathbf{u}_0) = 0, \tag{8}$$

$$\int d\mathbf{S} \cdot [\rho \mathbf{u}(\mathbf{u} - \mathbf{u}_0) + \mathbf{I}(p - p_0)] = -\mathbf{F}_d , \tag{9}$$

$$\int d\mathbf{S} \cdot \rho \mathbf{u}[(h - h_0) + \tfrac{1}{2}(u^2 - u_0^2)] = 0. \tag{10}$$

It is possible now to limit the surface integrations to that part of the surface S on which the ambient and actual properties differ, that is, the part lying within the nearly conical shock. Let us denote this part of S by the symbol S'.

3. First order integral relations. In the present section the earlier integral relations are specialized to the case in which on S' the deviation of actual from ambient properties is small in a certain sense. Several alternative requirements must be met for this situation to be achieved:

a. The body is sufficiently slender and pointed so that the deviation from ambient properties is small everywhere except perhaps in the boundary layer.
b. Alternatively, in the case of a blunt body, the surface S' must be sufficiently far downstream that the pressure has subsided nearly to ambient and that turbulent mixing can reduce the specific entropy close to ambient. Also, the stoichiometric composition must deviate negligibly from its ambient value on S'.

We assume that the incident ambient air is moving in the positive z-direction, i.e., $\mathbf{u}_0 = \mathbf{1}_z u_0$, and hence $\mathbf{F}_d = \mathbf{1}_z F_d$. For simplicity, let us require the surface S' to be the part of a plane perpendicular to the z axis that is bounded by the conical shock front (see Fig. 1). Thus, as indicated by the figure, the boundary of S' is a circle of radius R. The subscript 1 can be used to denote the deviation of any quantity from its ambient value, and any term that is linear in a deviation can be described as "first order." We consider averages over the surface S' defined by

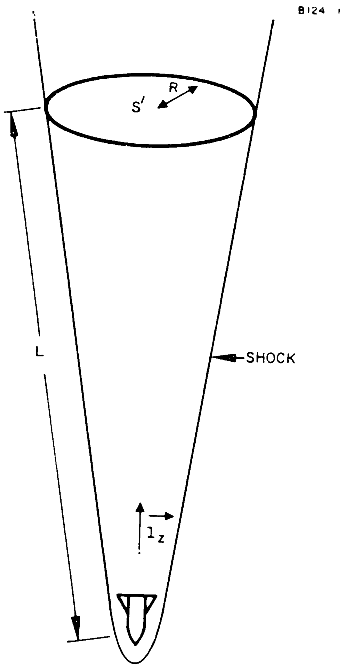

FIG. 1. Geometry of the wake and the surface S'.

$$\bar{g} = \frac{1}{A}\int dS g = \frac{1}{\pi R^2}\int_0^R gr\,dr, \tag{11}$$

where g is an arbitrary function of the radius r, $A(=\pi R^2)$ is the area of S', and the integration extends over S'. We choose as fundamental state variables the density ρ, the pressure p, the temperature T, and the velocity $\mathbf{u}$. Assuming perfect gas behavior, we can write

$$h = h(T), \tag{12}$$

$$p = \rho \mathcal{R} T/\mathfrak{M}, \tag{13}$$

where $\mathcal{R}$ is the gas constant (per mole), $\mathfrak{M}$ is the effective molecular weight of air, and T is the absolute temperature.

On the basis of the above assumptions and definitions, equations (8), (9), and (10), retaining only first order terms, reduce to

$$\rho_0 \bar{u}_{z1} + u_0 \bar{\rho}_1 = 0, \tag{14}$$

$$\rho_0 u_0 \bar{u}_{z1} + \bar{p}_1 = -F_d/A, \tag{15}$$

$$c_p \bar{T}_1 + u_0 \bar{u}_{z1} = 0, \tag{16}$$

where u_{z1} is the z-component of $\mathbf{u}_1$ and $c_p \equiv dh/dT$ is the specific heat at constant pressure. The above set of equations must be supplemented by the equation of state (13), which in first order and with surface averaging can be written

$$\bar{p}_1 = (\mathcal{R}/\mathfrak{M})(T_0 \bar{\rho}_1 + \rho_0 \bar{T}_1). \tag{17}$$

Equations (14), (15), (16), and (17) are a set of four linear equations with the four dependent variables $\bar{\rho}_1$, $\bar{p}_1$, $\bar{T}_1$, and $\bar{u}_{z1}$. Solution of this set yields

$$A\bar{\rho}_1 = 2\pi \int_0^R r\,dr\rho_1 = \frac{\gamma F_d}{u_0^2}\frac{1}{1 - M^{-2}}, \tag{18}$$

$$A\bar{p}_1 = 2\pi \int_0^R r\,dr p_1 = (\gamma - 1)F_d \frac{1 + [(\gamma - 1)M^2]^{-1}}{1 - M^{-2}}, \tag{19}$$

$$A\bar{T}_1 = 2\pi \int_0^R r\,dr T_1 = \frac{\mathfrak{M}(\gamma - 1)F_d}{\mathfrak{R}\rho_0}\frac{1}{1 - M^{-2}}, \tag{20}$$

$$A\bar{u}_{z1} = 2\pi \int_0^R r\,dr u_{z1} = -\frac{\gamma F_d}{\rho_0 u_0}\frac{1}{1 - M^{-2}}. \tag{21}$$

In the above expression γ is the ratio of specific heats at constant pressure and at constant volume, respectively, and $M = u_0(\mathfrak{M}/\gamma\mathfrak{R}T_0)^{1/2}$ is the free stream Mach number. These equations relate four different surface integrals in a simple manner to the drag force F_d .

In the limit of large Mach number ($M \gg 1$), the above results reduce to a simple form:

$$A\bar{\rho}_1 = \frac{\gamma F_d}{u_0^2}, \tag{22}$$

$$A\bar{p}_1 = (\gamma - 1)F_d, \tag{23}$$

$$A\bar{T}_1 = \frac{\mathfrak{M}(\gamma - 1)F_d}{\mathfrak{R}\rho_0}, \tag{24}$$

$$A\bar{u}_{z1} = -\frac{\gamma F_d}{\rho_0 u_0}. \tag{25}$$

The relative errors in the above expressions are of the order of M^{-2} and thus, for example, when $M = 10$, the relative error is about 1%.

It is interesting to rewrite the above results in terms of the relative changes from ambient values. We obtain (again in the limit of large M)

$$\frac{\bar{\rho}_1}{\rho_0} = -\frac{\bar{u}_{z1}}{u_0} = \frac{F_d}{M^2 p_0 A}, \tag{26}$$

$$\frac{\bar{p}_1}{p_0} = \frac{\bar{T}_1}{T_0} = \frac{(\gamma - 1)F_d}{p_0 A}. \tag{27}$$

It is of interest to note that $\bar{\rho}_1/\rho_0$ and $-\bar{u}_{z1}/u_0$ are smaller than $\bar{p}_1/p_0$ and $\bar{T}_1/T_0$ approximately by a factor $1/M^2$. However, it should be emphasized that $|\rho_1/\rho_0|$ or $|u_{z1}/u_0|$ at an arbitrary value of the radius R is not necessarily smaller than $\bar{p}_1/p_0$ by a factor of $1/M^2$. Leaving this qualification aside, equations (26) and (27) suggest that the main effect in the asymptotic part of a hypersonic wake is thermal, since the changes in density and velocity in a surface-average sense are negligible and the changes in pressure and temperature are equal in the same sense.

On the basis of the above remarks it is of interest to consider the total flow of excess heat through the surface S'. To the first order the deviation of the specific entropy from its ambient value is given by

$$T_0 s_1 = h_1 - \frac{p_1}{\rho_0} = \frac{\mathcal{R}\gamma T_1}{\mathfrak{M}(\gamma - 1)} - \frac{p_1}{\rho_0}. \tag{28}$$

Using (19) and (20) we find that the total flow of excess heat through S' is given to first order by

$$Q \equiv 2\pi \int_0^R r\, dr \rho_0 u_0 T_0 s_2 = A\rho_0 u_0 \left(\frac{\gamma}{(\gamma - 1)} \bar{T}_1 - \frac{1}{\rho_0} \bar{p}_1 \right) = F_d u_0 . \tag{29}$$

The result implies that in the asymptotic part of the wake the work performed in pulling the body against the drag force is converted in first order entirely into heat convected through S' and that the mechanical energy carried in the shock wave is of higher order. The shock wave in the neighborhood of a blunt body carries appreciable energy; however, it is converted into heat through dissipation at the shock front until the wave attains a lower amplitude in the larger asymptotic part of the wake.

4. Conditions of validity. The linearization involved in the derivation of the results of the last section is valid if the magnitudes of the ratios ρ_1/ρ_0 , p_1/p_0 , T_1/T_0 , and u_{z1}/u_0 are small compared with unity everywhere on the surface S'. A necessary (but not sufficient) condition for the validity of the linearization is that the surface averages of the above ratios (i.e., $\bar{\rho}_1/\rho_0$, etc.) be small compared with unity. Let us consider the condition

$$\bar{p}_1/p_0 \ll 1. \tag{30}$$

Use will be made of the well known expression for drag force

$$F_d = \tfrac{1}{2} C_D A^* \rho_0 u_0^2 , \tag{31}$$

where C_D is the drag coefficient of the body and A^* is its nominal frontal area. With the use of (27) and (31) it is possible to reduce (30) to the simple form

$$L/a \gg M^2, \tag{32}$$

where L is the distance between the body and the surface S' (see Fig. 1) and a is a characteristic radius of the body (e.g., $a = (C_D A^*/\pi)^{1/2}$). Since $L \cong MR$, the last condition expressed in terms of R reads

$$R/a \gg M. \tag{33}$$

Thus the surface S' must be at least at a distance $M^2 a$ from the body in order that the linearized theory of Section 3 be valid. Consideration of the smallness of the ratio $\bar{T}_1/T_0$ leads to identical results while the consideration of $\bar{\rho}_1/\rho_0$ and $\bar{u}_{z1}/u_0$ leads to a condition much weaker than (32).

Actually, in the case of blunt bodies, a distance very much larger than $M^2 a$ may be required for turbulent mixing to degrade the specific entropy to the neighborhood of its ambient value.