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THE INITIAL VIBRATIONS OF A SPINNING SHELL*

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Abstract. In this paper, we have studied the initial angular motion of a spinning shell with an overturning and yawing moment and shown that the frictional effects provided by the yawing moment change the initial rosette motion of the shell to one with a non-zero minimum yaw.

1. Introduction. It is well known that the initial angular oscillations of a spinning shell are very similar to those of the axis of a spinning top under gravity. If, as a first approximation, one assumes the center of gravity of the shell to move uniformly in a straight line and ignores the frictional damping forces, the angular motion of the axis of the top and that of the shell are identical, provided that (*i*) the top and the shell have the same axial spin and axial moment of inertia, (*ii*) the transverse moment of inertia of the top about its point of support equals the transverse moment of inertia of the shell about its center of gravity, and (*iii*) the moment of the force of gravity about the point of support of the top equals the moment of the aerodynamic forces on the shell about its center of gravity. To this degree of approximation, the formal solutions of the two problems are identical. The yawing motion of the spinning shell has been completely worked out in this way by Fowler, Gallop, Lock and Richmond [1], and Fowler and Lock [2] in two separate papers. These authors also conducted a series of experiments to study the nature of air forces acting on fairly stable and on slightly unstable projectiles and analysed the initial angular oscillations of these projectiles. Their experiments show that after the first half-period of oscillation the initial rosette motion of the shell axis changes to one with a non-zero minimum yaw. If, however, one solves the corresponding top problem in terms of elliptic functions, as has been done by these authors, the aforesaid angular motion remains far from being explained. This apparently unexplained motion is supposed by Fowler and Lock to be due to the dissipative effects produced by "other couples" depending on the angular velocity of the shell axis and also by the lateral motion of the center of gravity of the shell.

In this paper, we study the effects of damping on the initial yawing motion of the shell. As these effects are mainly due to the "yawing moment" we have taken this moment into consideration in addition to the tilting moment that tends to increase the yaw** of the shell. Treating the damping effects as weak***, we have obtained an approximate solution of the equation of yawing motion of the shell. This is in fact the γ

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**Yaw is the angle which the axis of the shell makes with the direction of motion of the center of gravity of the shell.

***Cf. [1], Sections 3.5 and 4.12.

type of equation of Fowler *et al.* corrected for the damping effects provided by the yawing moment. The "yawing moment" is thus seen to be responsible for changing the nature of the initial yawing motion of the shell from rosette motion to one with a non-zero minimum yaw. The minimum and maximum yaw attained by the shell during the initial oscillations of the shell axis have been calculated and shown to agree well with the observed values given by Fowler *et al.*

The solution we have obtained in this paper applies only to stable shells, where the yaw in the initial motion does not change considerably, since we have taken for the tilting moment an expression which corresponds to a slow yawing motion of the shell. For unstable shells, where the initial variations of yaw could be large, a Fourier expansion must be made up to two terms of the moment coefficients as has been done by Fowler and Lock. It is hoped that a solution of the corresponding problem for unstable shells with nonlinear tilting moment can be obtained in a similar manner. By comparison with the observed variation of yaw it should then be possible to determine the damping coefficient of the yawing moment for unstable shells.

2. Aerodynamic moments influencing motion. The complete aerodynamic force-system influencing the motion of a spinning shell consists of five forces and five couples resolved along and normal to the axis of the shell. The definition of these forces is given by Nielsen and Synge [3], and their complete vector specification by one of the present authors [4]. Out of the five air couples, the two that are supposed to influence the motion under study are (*i*) the tilting or overturning moment, tending to increase the yaw of the projectile and (*ii*) the yawing moment. The latter diminishes the cross angular velocity of the shell and thus provides a damping effect on the initial oscillation of the shell axis. The two air couples considered here are called by Nielsen and Synge "the cross torque due to cross velocity" and "the cross torque due to cross spin". According to [1] these torque vectors are given by

$$\mathbf{M} = \mu(\mathbf{k} \times \mathbf{\Lambda}), \tag{2.1}$$

$$\mathbf{H} = -hB(\mathbf{k} \times \mathbf{k}'), \tag{2.2}$$

where the prime indicates differentiation with respect to time, and $\mathbf{k}$ and $\mathbf{\Lambda}$ are unit vectors in the direction of the axis of the shell and parallel to the direction of motion of the centre of gravity. Other notations are as explained in [1]*. It is convenient to write

$$\mu = B\Omega^2/4s, \tag{2.3}$$

where

$$\Omega = A\omega_3/B. \tag{2.4}$$

Here, ω_3 is the axial spin of the shell and A and B are its axial and transverse moments of inertia (about the center of gravity). When the yaw does not exceed say 10°, the moment factor is independent of yaw and assumes the form (2.3).

3. Assumptions. (A.1) With Fowler and Lock we assume that the center of gravity of the shell moves uniformly in a straight horizontal line. This happens when the shell is fired horizontally with a large initial velocity, so that the analysis applies to the initial motions only. During this period the force due to gravity may be neglected.

*See [1], pp. 328–334.

(A.2) We also assume that the damping effect of the yawing moment is weak. This is in agreement with the observations of Fowler *et al.* As a result of this assumption we have

$$\epsilon = h/\Omega \ll 1. \tag{3.1}$$

(A.3) The axial spin ω_3 is constant during the considered motion.

4. Equations of angular motion. Let OA denote the direction of the axis and OP the direction of motion of the center of gravity of the shell. The angle of yaw then is $\angle AOP = \delta$. Let ϕ be the angle which the plane of yaw AOP makes with a fixed plane through OP. If the unit vectors along OA and OP be denoted by $\mathbf{k}$ and $\mathbf{\Lambda}$, the angular velocity $\boldsymbol{\omega}$ of the shell can be resolved according to $\boldsymbol{\omega} = \omega_3\mathbf{k} + \boldsymbol{\xi}$, where $\boldsymbol{\xi}$ is the cross angular velocity vector. Thus, $\mathbf{k}' = \boldsymbol{\xi} \times \mathbf{k}$ and $\boldsymbol{\xi} = \mathbf{k} \times \mathbf{k}'$, and the angular momentum is

$$\mathbf{H} = B\boldsymbol{\xi} + A\omega_3\mathbf{k} = B(\mathbf{k} \times \mathbf{k}') + A\omega_3\mathbf{k}.$$

If $\mathbf{G}$ is the torque due to the air forces, we have

$$\frac{d\mathbf{H}}{dt} = \mathbf{G},$$

i.e.

$$\begin{aligned} B(\mathbf{k} \times \mathbf{k}'') + A\omega_3'\mathbf{k} + A\omega_3\mathbf{k}' &= \mathbf{G}, \\ B(\mathbf{k} \times \mathbf{k}'') + A\omega_3\mathbf{k}' &= \mathbf{G}, \end{aligned} \tag{4.1}$$

where

$$\mathbf{G} = \mu(\mathbf{\Lambda} \times \mathbf{k}) - hB(\mathbf{k} \times \mathbf{k}').$$

According to (A.1), we may write

$$\begin{aligned} \mathbf{\Lambda} &= (1, 0, 0), \qquad \mathbf{k} = (l, m, n), \\ \mathbf{k}' &= (l', m', n'), \qquad \mathbf{k}'' = (l'', m'', n''). \end{aligned} \tag{4.2}$$

In these relations (l, m, n) are the direction-cosines of the shell axis with respect to a set of fixed coordinate axes $OXYZ$, where OXY is the plane of fire and the axis OZ is to the right of the gunner, so that one has

$$l = \cos\delta, \qquad m = \sin\delta\cos\phi, \qquad n = \sin\delta\sin\phi. \tag{4.3}$$

Projecting the vector equation (4.1) along the vectors $\mathbf{\Lambda}$ and $\mathbf{k} \times \mathbf{k}'$ and making use of Eqs. (4.2) and (4.3), we have

$$(\Omega\cos\delta + \phi'\sin^2\delta)' + h\phi'\sin^2\delta = 0, \tag{4.4}$$

$$\tfrac{1}{2}(\delta'^2 + \phi'^2\sin^2\delta)' + h(\delta'^2 + \phi'^2\sin^2\delta) = -(\mu/B)(\cos\delta)'. \tag{4.5}$$

If we put $h = 0$ in these equations (i.e. when there is no damping effect due to a yawing moment), we obtain after integration

$$\phi'\sin^2\delta + \Omega\cos\delta = E, \tag{4.6}$$

$$(\delta'^2 + \phi'^2\sin^2\delta) + \int (2\mu/B)\, d(\cos\delta) = F, \tag{4.7}$$

where E and F are constants of integration. These are precisely the equations considered by Fowler and Lock. When μ is constant, they represent the integrals of energy and angular momentum equations of an equivalent spinning top.

Our aim is to study Eqs. (4.4) and (4.5) for a constant μ.

5. The nonlinear equation in yaw. With

$$c = \cos \delta, \qquad (\mu/B) = (\Omega^2/4s) \tag{5.1}$$

Eqs. (4.4) and (4.5) can be written as

$$(\phi' \sin^2 \delta)' + h(\phi' \sin^2 \delta) = -(\Omega c'),$$

$$(\delta'^2 + \phi'^2 \sin^2 \delta)' + 2h(\delta'^2 + \phi'^2 \sin^2 \delta) = -(\Omega^2/2s)c'.$$

On integration they give

$$(\phi' \sin^2 \delta) \exp (ht) = -\Omega \int c' \exp (ht)\, dt + E,$$

$$(\delta'^2 + \phi'^2 \sin^2 \delta) \exp (2ht) = -(\Omega^2/2s) \int c' \exp (2ht)\, dt + F,$$

where E and F are constants of integration to be determined from the initial conditions.

Eliminating ϕ' between the preceding equations, one obtains the equation of yaw in the form

$$c'^2 \exp (2ht) + \left\{E - \Omega \int c' \exp (ht)\, dt\right\}^2 + (1 - c^2)\left\{(\Omega^2/2s) \int c' \exp (2ht)\, dt - F\right\} = 0. \tag{5.2}$$

Differentiating (5.2) three times with respect to t, we have

$$(c' - hc)\left\{c^{\text{iv}} + 2hc''' + (\Omega^2 + h^2)c'' - \frac{\Omega^2}{s}(cc'' + c'^2) - \frac{\Omega^2}{2s} hcc'\right\} - (c'' - 3hc' + 2h^2c) \cdot \left\{c''' + 2hc'' + (\Omega^2 + h^2)c' - \frac{\Omega^2}{s} cc' + \frac{\Omega^2}{4s} h(1 - c^2)\right\} = \frac{\Omega^2}{2s} c'(c' - hc).$$

In this equation we write

$$\tau = \Omega t \tag{5.3}$$

to obtain, with some re-arrangements of terms,

$$\sum_{\nu=0}^{4} \epsilon^\nu F_\nu = 0, \tag{5.4}$$

where*

$$F_0 = c'c^{\text{iv}} - c''c''' - (3/2s)c'^3, \tag{5.5}$$

$$F_1 = -cc^{\text{iv}} + 5c'c''' - c''\left\{2c'' + \frac{1 - 5c^2}{4s} + c\right\} - 3c'^2(c/2s - 1), \tag{5.6}$$

$$F_2 = -4cc''' + 6c'c'' - c'\left\{2c - \frac{1}{4s}(3 + 5c^2)\right\}, \tag{5.7}$$

*Prime now denotes differentiation with respect to τ.

$$F_3 = -5cc'' + 3c'^2 - (1 - c^2)c/2s, \tag{5.8}$$

$$F_4 = -2cc'. \tag{5.9}$$

Equation (5.4) describes the yawing motion of the shell.

6. A sinusoidal approximation to the undamped motion in yaw. It follows from the preceding analysis that the undamped motion in yaw is given by

$$c'c^{\text{iv}} - c''c''' = (3/2s)c'^3. \tag{6.1}$$

Assuming an initial rosette motion with Fowler and Lock, we stipulate the initial conditions as

$$c = 1; \qquad c' = 0; \qquad c'' = -b^2 \quad \text{at} \quad \tau = 0, \tag{6.2}$$

where b measures the size of the initial disturbances which upset the nose-on motion of the shell. This is of small magnitude say, of the order 10^{-3}.

If we set

$$(c'''/c') = \frac{1}{s} - 1 - b^2, \tag{6.3}$$

and use conditions (6.2), Eq. (6.1) reduces after three integrations to the well known equation of a top

$$c'^2 + (1 - c)^2 + (1 - c^2)(c/2s - b^2 - 1/2s) = 0, \tag{6.4}$$

an equation that would have normally followed from (4.6) and (4.7) by eliminating ϕ' between them and subjecting them to the initial conditions.

A solution of (6.4) in terms of elliptic functions is given by

$$c = 1 - 2\sin(\Delta/2)cn(K - \psi\tau, k), \tag{6.5}$$

where $\Delta = \max \delta$, K is the complete elliptic integral of the first kind of modulus k and the factor ψ is given by

$$\psi = \frac{\sin(\Delta/2)}{2k(s)^{1/2}} \tag{6.6}$$

A sinusoidal approximation to (6.5) is

$$c = \alpha + \beta \cos \lambda\tau, \tag{6.7}$$

where

$$\alpha + \beta = 1, \qquad \beta = \sin^2 \Delta/2, \qquad \lambda^2 = b^2/\beta + \beta/2s. \tag{6.8}$$

This may be proved as follows.* The general solution of (6.4) may be written as

$$\tau = \int \frac{dc}{\frac{1}{2s}\{(\alpha_1 - c)(c - 1)(c - \alpha_2)\}^{1/2}}, \tag{6.9}$$

where

$$\alpha_2 > 1 > \alpha_1, \qquad \alpha_1 + \alpha_2 = 2s(1 + b^2), \qquad \alpha_1\alpha_2 = -(1 - 2s + 2sb^2). \tag{6.10}$$

*Cf. [4], pp. 284–285.

Since, throughout the motion, $\alpha_1 < c < 1$ and therefore $\alpha_2 - 1 < \alpha_2 - c < \alpha_2 - \alpha_1$, it follows from (6.9) that

$$\left\{\frac{\alpha_2 - 1}{2s}\right\}^{-1/2} \cos^{-1} \kappa > \tau + \text{const.}, > \left\{\frac{\alpha_2 - \alpha_1}{2s}\right\}^{-1/2} \cos^{-1} \kappa,$$

where

$$a = \frac{1 - \alpha_1}{2}, \qquad \kappa = \frac{c}{a} - \frac{1 + \alpha_1}{2a}.$$

If α_1 comes close enough to unity, which happens when the shell spins fast enough to be stable, and if we set $1 + \alpha_1 = 2\sigma$, we obtain

$$\tau + \text{const} = \left\{\frac{\alpha_2 - \sigma}{2s}\right\}^{-1/2} \cos^{-1} \left\{\frac{2(c - \sigma)}{1 - \alpha_1}\right\},$$

which under the initial conditions reduces to (6.7).

7. An approximate solution of the yaw equation with damping. Consider the equation (5.4). We assume an asymptotic solution of this in the form

$$c = \alpha + \beta \cos \lambda\tau + \epsilon c_1(\tau), \tag{7.1}$$

since β and ϵ are much less than unity. When substituted in (5.4) this yields

$$\beta\epsilon(L) + \epsilon^2(M) + \cdots = 0, \tag{7.2}$$

where

$$L = -\lambda \sin \lambda\tau(c_1^{\text{iv}} + \lambda^2 c_1'') + \lambda^2 \cos \lambda\tau(c_1''' + \lambda^2 c_1' + K), \tag{7.3}$$

$$M = c_1' c_1^{\text{iv}} - c_1'' c_1''' - \alpha c_1^{\text{iv}} - c_1''(K + \alpha\lambda^2), \tag{7.4}$$

$$K = \frac{1}{4s}(1 - 5\alpha^2) + \alpha(1 - \lambda^2). \tag{7.5}$$

Terms of orders in excess of 2 in β and ϵ are not shown in (7.2), since we shall not be needing them for our solution.

It is not difficult to see that

$$L = M = 0 \tag{7.6}$$

if

$$c_1^{\text{iv}} + \lambda^2 c_1'' = 0, \tag{7.7}$$

$$c_1''' + \lambda^2 c_1' + K = 0. \tag{7.8}$$

In fact, (7.7) is obtained by differentiating (7.8) with respect to τ. The initial conditions, as stipulated by Fowler, are

$$\delta = 0, \qquad \delta' = b \quad \text{when} \quad \tau = 0.$$

From these we have

$$c_1'(0) = 0, \qquad c_1''(0) = \beta\lambda^2 - b^2. \tag{7.9}$$

Solving (7.8) with the use of (7.9), we get as approximate solution of the equation of yaw (5.4)

$$c = (1 - b^2/\lambda^2) + (b^2/\lambda^2) \cos \lambda\tau - (K\epsilon/\lambda^3)(\lambda\tau - \sin \lambda\tau). \tag{7.10}$$

Subsequent calculations show that the constant K is of extremely small (positive) magnitude. For stable shells it turns out to be roughly of the order 10^{-4}.

8. The nonzero minimum yaw. If d_n and D_n denote the minimum and maximum values the yaw assumes in the nth half-period of vibration of the shell axis, a straightforward calculation from (7.10) yields

$$\sin^2 (d_n/2) = n\pi K\epsilon/\lambda^3, \tag{8.1}$$

$$\sin^2 (D_n/2) = b^2/\lambda^2 + (K\epsilon/\lambda^3)\left(n\pi - \tan^{-1} \frac{b^2\lambda}{K\epsilon}\right). \tag{8.2}$$

The time period between two successive minima or maxima is given by $2\pi/\lambda\Omega$. This shows the frequency of oscillations of the shell axis is the same for the damped and the undamped motion characterised by the approximate solutions (7.10) and (6.8). This is not strictly true. But for stable shells, it gives a fairly accurate result. Also from (8.1) it is clear that for $K > 0$ subsequent minimum values of yaw go on increasing steadily starting from a zero initial value.

For the axis of the shell to come to a position of equilibrium yaw, after the initial vibrations are damped out, the maximum yaw given by (8.1) must decrease steadily with increasing time. But this does not follow from (8.1); on the contrary we observe that the maximum yaw increases. This shows that the solution (7.10) cannot be continued too far as is obvious from the presence of a secular term in it. The first nonzero minimum yaw predicted by (7.10) agrees however well with observation, as is seen from Table I.

9. Concluding remarks. The present solution does not work for unstable or just stable shells. In the latter case k turns out to be negative and hence no minimum yaw can be correctly predicted. For unstable shells, where the initial variation of yaw is relatively large, say, with an initial maximum of 15°, the calculated variation of yaw

TABLE 1

Table showing the observed and calculated values of maximum and minimum yaw of a spinning shell during the 1st half period of vibration of the shell axis.

(1): No. of the group and rounds fired
(2): Stability factor $s = \beta\Omega^2/4\mu$
(3): $\lambda^2 = b^2/\beta + \beta/2s$
(4): $\beta = \sin^2(\Delta/2)$ ($10^3\beta$ is tabulated)
(5): $k = (1 - 5\alpha^2)/4s + \alpha(1 - \lambda^2)$ ($10^3\,k$ is tabulated)
(6): $\epsilon = h/\Omega$
(7) & (8): calculated values of the first minimum d_1 and maximum D_1 (in degrees) using formulae (8.1) and (8.2)
(9) & (10): observed values of d_1 and D_1 (in degrees)

1	2	3	4	5	6	7	8	9	10
I–24	1.61	0.379348	0.675	0.163	0.0150	0.656	3.0	+0.4	2.8
I–28	1.74	0.425892	0.848	0.126	0.0102	0.45	3.35	−0.4	3.1
II–23	1.94	0.484836	0.404	0.0130	0.0121	0.1	2.3	+0.3	2.3
III–22	1.84	0.456983	0.632	0.0545	0.0104	0.266	2.9	−0.1	3.5
IV–25	1.525	0.344701	0.653	0.204	0.0150	0.8	2.9	+0.0	2.2

is not in good agreement with observation. To predict the initial variation of yaw in such cases, the dependence of the factor μ of the tilting moment on yaw must be treated more precisely as has been done by Fowler and Lock. If the corresponding problem for unstable shells can be solved, this will make possible a quantitative estimate of the damping factor provided by the yawing moment. The authors hope to solve this non-linear problem by a method essentially similar to that of the present paper.

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