A NONLINEAR THEORY OF PLASTICITY FOR PLANE STRAIN*

BY

E. M. SHOEMAKER**

Sandia Laboratory, Albuquerque, New Mexico

Summary. A nonlinear theory of plasticity is proposed which facilitates the solution of a restricted class of plane strain problems. An anisotropic approximation to a real material is utilized with the characteristic directions chosen a priori. The formulation is in terms of displacements and sufficient displacement boundary conditions must be prescribed in order to solve the resulting classical wave equation. Subsequently, the stress field is determined, corresponding to any nonlinear monotonic relation between shear stress and strain, by direct integration of the equilibrium equations. Uniqueness restricts the class of problems suitable to the theory. Classical isotropic theory will be closely approximated in problems where the deviatoric stress field is predominantly uniform. The theory is illustrated by the solution of an indentation problem.

1. Introduction. The idea of approximating the ellipsoidal cylinder which is the von Mises yield surface for plane strain by a piecewise linear yield surface has been previously applied by the writer [7]. This formulation utilized the four planes shown in Fig. 1 although other approximations are equally acceptable. The theory was aimed at the limit load problem of perfect plasticity and resulted in a linearization of the field equations. Onat and Prager [2] previously applied a linearization procedure to solve the problem of a tension specimen in plane flow.

One difficulty which was not surmounted is inherent in any rigid-plastic or elastic-plastic theory and results from the fact that the material behavior must be characterized by two expressions. The inevitable consequence is two distinct sets of field equations whose corresponding domains cannot in general be determined analytically. The same difficulty is encountered in the elastic-plastic torsion problem where it is more easily delineated (Prager and Hodge [1]). It has long been recognized that this difficulty is removed if the material behavior is characterized by a single expression. Unfortunately, such a formulation for a stress-type boundary-value problem will always be highly nonlinear.

This paper develops a nonlinear deformation theory of plasticity for plane strain which approximates an isotropic hardening von Mises loading surface by an isotropic hardening piecewise linear loading surface, Fig. 1. Plastic incompressibility is inherent in the choice of the loading surface. Elastic dilatational strain is neglected and the elastic deviatoric strain is restricted to be codirectional with the plastic strain. In common with all deformation theories, unloading is ruled out; consequently, the theory reduces to a nonlinear incompressible elastic theory. The stress-strain law is assumed to be monotonically increasing but otherwise arbitrary; although, for convenience, this relation should be expressible in terms of a single analytic expression whose inverse may be found explicitly. The present development is restricted to small displacements.

*Received March 25, 1965; revised manuscript received July 15, 1965. This work was supported by the United States Atomic Energy Commission.

**Present Address: Department of Mathematics, Simon Fraser University, Vancouver, B. C., Canada. This work was performed while the author served as a Visiting Summer Professor at Sandia Laboratory.

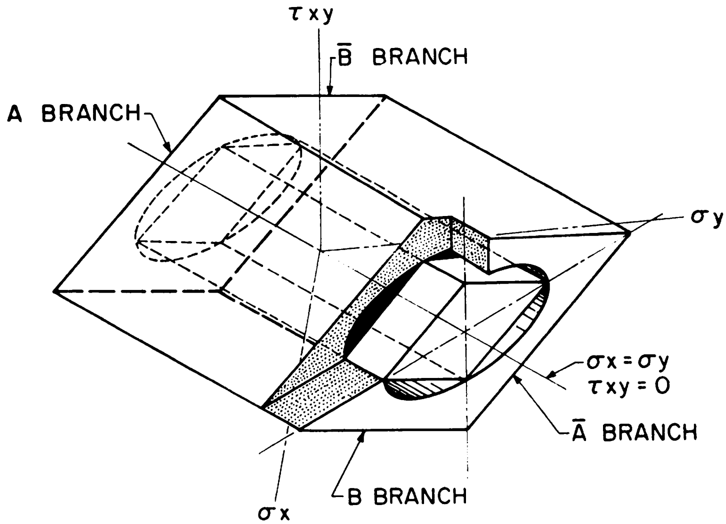

FIG. 1. von Mises yield ellipsoidal cylinder with inscribed piecewise linear yield surface and isotropically expanded loading surface.

The theory is formulated in terms of displacements and its application is restricted to problems where sufficient displacement-type boundary conditions are known in order to determine the displacement field independent of the stress field. (It will be seen that because the displacement equations are hyperbolic the boundary conditions must contain both stress-type and displacement-type conditions.)

This development was motivated by the need to determine the pressure distribution on the contact faces of pistons used in high pressure research investigation (Balchan and Drickamer [6]), where the permanent displacements at the boundary are easily measured. Other problems which appear to be amenable include pressing between rigid dies, extrusion, and certain problems of indentation, one of which is used to illustrate the theory.

2. General development. In applications of the theory we shall assume that the stress field lies on a single branch of the loading surface. (This is not a general restriction as will be seen in the discussion of uniqueness. However, the more general problem is of a higher order of difficulty. Also, under this restriction the theory will be shown to be equivalent to an incremental theory.) This assumption places restrictions upon the displacement boundary conditions which will be discussed. As in the writer's perfectly-plastic theory [7] the displacement field is hyperbolic and the characteristic coordinates are rotated counterclockwise through a fixed angle β with respect to the Cartesian system (x, y) which may be chosen to approximate some anisotropic material behavior but is otherwise arbitrary. The system (x, y) can vary continuously in direction throughout the body but will have fixed directions in the present development. Corresponding to the A, $\bar{A}$ and $(B, \bar{B})$ branches of Fig. 1, $\beta = -\pi/8$ $(+\pi/8)$.

Consider a particular branch of the loading surface and let (ξ, η) be the characteristic coordinates and (u, v) the corresponding displacements. Since the characteristic directions are directions of pure shear, the shear strain is given by

$$\gamma_{\xi\eta} = \frac{\partial u}{\partial \eta} + \frac{\partial v}{\partial \xi} \tag{1}$$

whereas the normal strains vanish. Consequently,

$$\frac{\partial u}{\partial \xi} = 0, \qquad \frac{\partial v}{\partial \eta} = 0, \tag{2}$$

and integration immediately renders the general solution

$$u(\xi, \eta) = f(\eta), \tag{3}$$

$$v(\xi, \eta) = g(\xi). \tag{4}$$

From Eq. (1) the shear strain is given by

$$\gamma_{\xi\eta} = f'(\eta) + g'(\xi). \tag{5}$$

It is clear that displacement boundary conditions which are sufficient to determine the displacement field are very similar to conditions required by the classical string equation. The fundamental problems are the Cauchy problem and the Riemann problem pertaining to open regions. Obviously, assigning the displacement vector over a closed boundary is to overprescribe the side conditions.

The shear stress $\tau_{\xi\eta}$ is determined immediately through some suitable monotonic stress-strain relationship. An example would be the empirical law

$$\gamma_{\xi\eta} = \tau_{\xi\eta}/G + K\tau_{\xi\eta}^{1/n} \tag{6}$$

which closely approximates the behavior of most work hardening metals. Prager's law,

$$\tau_{\xi\eta} = Y \tanh\left(\frac{3G}{Y}\gamma_{\xi\eta}\right) \tag{7}$$

in which Y is the limiting shear stress, expresses the stress explicitly in terms of strain and approximates the behavior of a perfectly plastic material.

Assuming $\tau_{\xi\eta}$ known, the normal stresses σ_ξ, σ_η are obtained through integration along the respective characteristics of the equilibrium equations

$$\frac{d\sigma_\xi}{d\xi} = -\frac{\partial \tau_{\xi\eta}}{\partial \eta}, \tag{8}$$

$$\frac{d\sigma_\eta}{d\eta} = -\frac{\partial \tau_{\xi\eta}}{\partial \xi}. \tag{9}$$

Sufficient stress-type boundary data must be prescribed in order to determine the initial conditions.

Because the displacement field is the solution of a hyperbolic equation, any discontinuities in boundary displacements will propagate into the interior along the characteristics as can be seen from Eqs. (3) and (4). Displacement boundary data must be prescribed so as to rule out this possibility. Continuity of boundary displacements is a necessary but not a sufficient condition to insure this condition as will be illustrated by a later example.

Discontinuities in the derivatives of boundary displacements lead to discontinuities in shear stress $\tau_{\xi\eta}$ across characteristics, from Eq. (5). Consequently, in order to satisfy equilibrium, boundary displacements must be at least smooth. There still remains the possibility for the usual stress discontinuities (Hill [5]).

3. Uniqueness. It is possible to establish uniqueness for the mixed boundary-value problem sketched in the previous section. (We do not limit the stress vector to lie on a single branch of the loading surface.) Consider a finite body where at each point on the boundary either the surface traction vector, the displacement vector, or the tangential component of one and the normal component of the other are defined. The uniqueness proof proceeds in the usual fashion (Hill [4]). Assume that two solutions to such a problem exist. Since displacement discontinuities are ruled out, the usual application of the principle of virtual work renders

$$\int_V (\sigma_{ij}^{(1)} - \sigma_{ij}^{(2)})(\epsilon_{ij}^{(1)} - \epsilon_{ij}^{(2)})\, dV = 0. \tag{10}$$

For the present case this becomes

$$\int_V (s_i^{(1)} - s_i^{(2)})(e_i^{(1)} - e_i^{(2)})\, dV = 0, \tag{11}$$

where s_i and e_i are defined in Fig. 2.

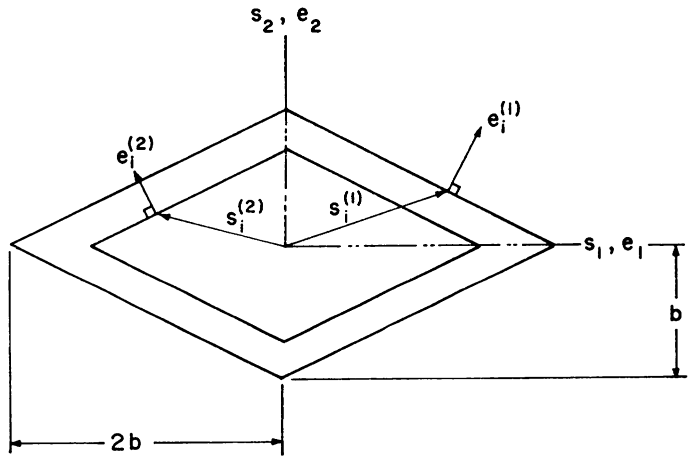

FIG. 2. Geometric representation of stress and strain deviators with isotropic hardening linearized loading locus. For the case shown, $|e_i^{(1)}| > |e_i^{(2)}|$; $s_1 = (\sigma_y - \sigma_x)$, $s_2 = \tau_{xy}$, $e_1 = (\epsilon_y - \epsilon_x)$, $e_2 = 2\gamma_{xy}$ with $\epsilon_x + \epsilon_y = 0$.

The stress-strain relation in this (x, y) coordinate system, as opposed to the characteristic coordinates of Eqs. (6) and (7), is conveniently written as

$$e_i = h(|s_1| + 2\,|s_2|)n_i \tag{12}$$

with h a positive monotonic increasing scalar function and n_i the unit outer normal to any of the four branches of any loading surface of Fig. 2. (In general, at a corner, n_i can have any direction in the fan angle between the outer normals to the adjoining branches.) We note the assumption that the loading surface expands isotropically; Eq. (12) also assumes the strain magnitude to be dependent upon the hardening function h but otherwise independent of the strain direction.

Substitution of Eq. (12) into Eq. (11) establishes that the integrand is positive except for two cases. The first, when $s_i^{(1)} = s_i^{(2)}$ implies that $e_i^{(1)} = e_i^{(2)}$ except at corners. Uniqueness of the characteristic shear stress $\tau_{\xi\eta}$ follows from uniqueness of s_i even at corners where the orientation of the characteristic coordinates is arbitrary. Moreover, if we restrict the direction of e_i at corners to be normal to either side the strain energy is unique. Secondly, the integrand of (11) is zero when $e_i^{(1)} = e_i^{(2)}$. Uniqueness of $\tau_{\xi\eta}$ as well as the strain energy follows for this case.

Uniqueness of normal stresses σ_ξ and σ_η is not obtained by this development but follows directly from the integration of Eqs. (8) and (9), along with the fact that $\tau_{\xi\eta}$ is unique, provided that in the particular problem considered one and only one initial condition is prescribed along the characteristic.

These results depend upon the assumption of isotropic hardening. The integrand of Eq. (11) can be negative if the four branches of the loading surface are permitted to expand independently as in Fig. 3 or as in slip theory (Batdorf and Budiansky [3]).

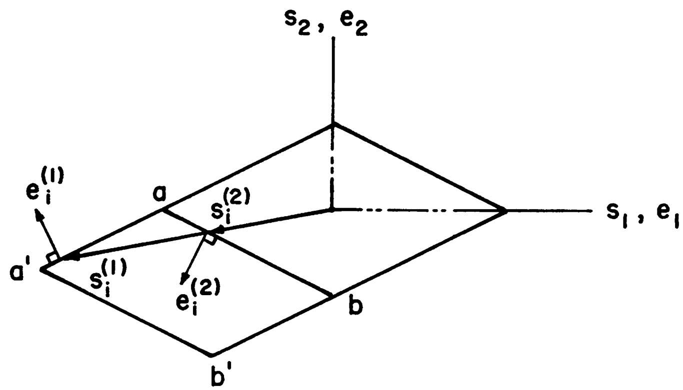

FIG. 3. Anisotropic hardening leads to nonuniqueness. Here, independent hardening of each branch, e.g., a translation of side ab into $a'b'$, gives $(s_i^{(1)} - s_i^{(2)})(e_i^{(1)} - e_i^{(2)}) < 0$. With no hardening on remaining three branches, $|e_i^{(1)}| = |e_i^{(2)}|$ for the case shown.

Clearly, this deformation theory is equivalent to an incremental theory provided that in the solution of a given problem by the corresponding incremental theory some loading sequence exists such that at each point in the material the same loading branch applied throughout the loading process. In particular, for a problem where only one branch is utilized in the deformation solution, it will generally be possible to show that the same branch can apply throughout the loading process of the corresponding incremental theory. (This is illustrated in the following example.)

The following example will emphasize that the mixed boundary conditions must be prescribed in certain patterns if a solution is to exist. If, in the example, any stress-type condition were to be replaced by a displacement condition the displacement problem would be overprescribed.

4. Indentation of a slab. Fig. 4 shows a rectangular section slab, resting upon a rigid frictionless surface, being indented from above. The characteristic coordinates (ξ, η) are oriented at 45° with the tacit assumption that the particular boundary conditions imposed will make it possible to utilize only one branch of the loading surface.

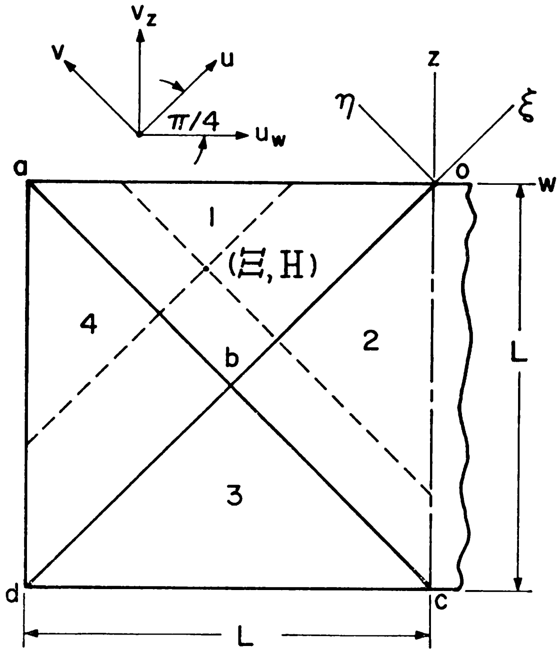

FIG. 4. Rectangular slab showing characteristics through a point (Ξ, H), coordinate systems and displacement definitions.

We first subtract off a uniform negative shear $\gamma_{\xi\eta} = -\delta$ of undetermined magnitude (to be added later) which corresponds to a uniform contraction in the z direction and equal elongation in the w direction. With oc a line of symmetry, the boundary conditions on three sides take the form:

on oc:

$$u_w(0, z) = 0, \tag{13}$$

$$\tau_{wz}(0, z) = 0; \tag{14}$$

on cd:

$$v_z(w, -L) = 0, \tag{15}$$

$$\tau_{wz}(w, -L) = 0; \tag{16}$$

on ad:

$$\sigma_w(-L, z) = 0, \tag{17}$$

$$\tau_{wz}(-L, z) = 0. \tag{18}$$

On side oa we choose the displacement conditions

$$v_z(w, 0) = -A\left[\cos\left(\frac{\pi w}{L}\right) + 1\right] - C, \tag{19}$$

$$u_w(w, 0) = -B\left[-\cos\left(\frac{\pi w}{L}\right) + 1\right] - D, \tag{20}$$

with constants A and B positive, as being representative of displacements which might be caused by an indenter.

Eqs. (19) and (20) are Cauchy conditions for region 1. Eqs. (3) and (4) determine the functions $f_1(\eta)$, $g_1(\xi)$. Continuity of displacement v across ob and Eq. (13) constitute a type-three problem in region 2. Note that $g_2(\xi) = g_1(\xi)$. Continuity of u across bc and Eq. (15) determine the displacement field in region 3 with $f_3(\eta) = f_2(\eta)$. Finally, continuity of u across ab and of v across bd represents a Riemann problem in region 4 with $f_4(\eta) = f_1(\eta)$ and $g_4(\xi) = g_3(\xi)$.

We note that continuity of tangential displacement across ob requires that $D = 0$; displacement continuity across bc requires that $C = 2B$. This illustrates our previous remark that prescribing continuous boundary displacements is not sufficient to insure a continuous field.

The resulting shear strain field is

$$\gamma_{\xi\eta} = \frac{\pi}{L}\left[(A - B)\sin\left(\frac{\sqrt{2}\pi\eta}{L}\right) - (A + B)\left|\sin\left(\frac{\sqrt{2}\pi\xi}{L}\right)\right|\right] \tag{21}$$

in regions 1 and 4, and

$$\gamma_{\xi\eta} = \frac{\pi}{L}(A + B)\left[\sin\left(\frac{\sqrt{2}\pi\eta}{L}\right) - \left|\sin\left(\frac{\sqrt{2}\pi\xi}{L}\right)\right|\right] \tag{22}$$

in regions 2 and 3. Note that $\gamma_{\xi\eta}$ is continuous but the derivatives are not continuous on od and ac.

Eqs. (21) and (22) do not as yet represent a unique strain field. The uniqueness proof assumes isotropic hardening. The solution assumes that the stress vector lies on a single branch of the loading surface. For these two assumptions to be consistent in this example the stress field must satisfy the consistency conditions:

$$\tau_{\xi\eta} \leq 0, \tag{23}$$

$$|2\tau_{\xi\eta}/(\sigma_\xi - \sigma_\eta)| \geq 1. \tag{24}$$

Condition (23) is clearly satisfied by Eqs. (21) and (22) if $B \geq A \geq 0$. Condition (24) is not satisfied a priori; the stress field must first be determined. If, for a particular choice of constants A and B and a particular stress-strain equation, condition (24) were not satisfied everywhere it must be concluded that the same branch does not hold everywhere. This presents a difficult problem which will not be examined here.

We consider the stress field for the case $A = B$ and make the substitutions $\xi' = \xi/L$, $\eta' = \eta/L$, $e = 2\pi GA/YL$, $\tau' = \tau/Y$, $\gamma' = G\gamma/Y$. The strain field becomes

$$\gamma'_{\xi'\eta'} = -e\,|\sin(\sqrt{2}\pi\xi')| - \delta' \tag{25}$$

in regions 1 and 4, and

$$\gamma'_{\xi'\eta'} = e[\sin(\sqrt{2}\pi\eta') - |\sin(\sqrt{2}\pi\xi')|] - \delta' \tag{26}$$

in regions 2 and 3, where $\delta' = G\delta/Y$ is the uniform shear strain (previously subtracted) which has been added to the solution.

Regardless of the particular stress-strain equation chosen, the method of determining the stress is the same in principle. The stress state on ad is known. This provides the initial conditions for the integration of Eq. (8) along $\eta =$ (const) lines and of Eq. (9)

along ξ = (const) lines intersecting ad. This determines $\sigma_{\xi'}$ in regions 1 and 4 and $\sigma_{\eta'}$ in regions 3 and 4. With the stress state on cd now known, integration of Eq. (8) provides $\sigma_{\xi'}$ in regions 2 and 3. Finally, integration of Eq. (9) with the initial conditions now known on oc determines $\sigma_{\eta'}$ in regions 1 and 2. It remains only to see if the stress field satisfies the consistency relation (24). If so, the solution is unique. This relation will always be satisfied provided the constant strain δ' is chosen to be large enough.

For linear behavior, $\tau = G\gamma$ or $\tau' = \gamma'$, the integration may be rendered in closed form. The results are:

$$\sigma'_{\xi'} = -e \sin(\sqrt{2}\pi\eta') - \delta' \tag{27}$$

in regions 1 and 4,

$$\sigma'_{\eta'} = e[-\sqrt{2}\pi(\eta' - \xi' - \sqrt{2}) \cos(\sqrt{2}\pi\xi') - \sin(\sqrt{2}\pi\xi')] - \delta' \tag{28}$$

in regions 3 and 4,

$$\sigma'_{\xi'} = e[\sqrt{2}\pi(-\xi' - 3\eta' - \sqrt{2}) \cos(\sqrt{2}\pi\eta') + \sin(\sqrt{2}\pi\eta')] - \delta' \tag{29}$$

in regions 2 and 3, and

$$\sigma'_{\eta'} = e[\sqrt{2}\pi(-3\xi' - \eta' - \sqrt{2}) \cos(\sqrt{2}\pi\xi') + \sin(\sqrt{2}\pi\xi')] - \delta' \tag{30}$$

in regions 1 and 2. Lines od and ac are discontinuity lines of the interior normal stress.

Investigation of the consistency relation (24) discloses that it is not satisfied for all values of δ'. The limitation is that $\delta' \geq \pi e$ or $\delta \geq 2\pi^2 A/L$.

For purposes of illustrating the application of a nonlinear stress-strain relation, the numerical solution corresponding to Prager's law, Eq. (7), was undertaken. Fig. 5 illustrates the results. For all cases we chose $\delta' = \pi e$ for comparison with the linear case.

For small values of the indentation parameter, $e < 0.01$, the nonlinear stress-strain equation is linear to three places. As a check, the case $e = 0.01$ was solved numerically and agreed with the previous linear solution to two places. The results for increasing

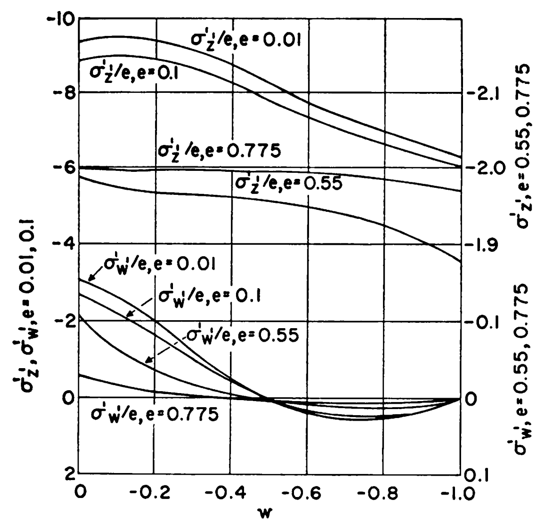

FIG. 5. Dimensionless surface tractions $\sigma'_{z'}/e$, $\tau'_{w'z'}/e = \sigma'_{w'}/e$ at indented surface. Linear behavior applies for $e < 0.01$. Uniform stress field $\sigma_z = -2.0Y$ applies for $e > 1.0$.

values of the indentation parameter e clearly show that the stress field approaches the uniform limit stress $\sigma_z = -2Y$, $\sigma_w = \tau_{wz} = 0$. For $e = 1.0$ this limiting state is reached to within two significant figures. (Since $e = 2\pi GA/YL$, using properties of mild steel, $e \simeq 7500\ A/L$. Consequently, the condition $e = 1.0$ would easily satisfy the small strain assumption.)

In all examples the consistency relation (24) achieved its minimum value at the point (0, 0) in region 1, Fig. 4. These values were 1.0, 1.1, 8.5, 31.6, 124, corresponding, respectively, to $e = 0.01, 0.1, 0.55, 0.775, 1.0$. The same branch is clearly applicable throughout the loading process.

Acknowledgement. The author is indebted to A. Iacoletti for assistance in obtaining the numerical results.

References

1. W. Prager and P. G. Hodge, *Theory of perfectly plastic solids*, Wiley, New York, 1951
2. E. T. Onat and W. Prager, *The necking of a tension specimen in plane plastic flow*, J. of Applied Physics **25** (1954)
3. S. B. Batdorf and B. Budiansky, *Polyaxial stress-strain relations of strain-hardening metal*, J. of Applied Mech. **21** (1954)
4. R. Hill, *On the problem of uniqueness in the theory of a rigid-plastic solid.* I, J. of the Mech. and Physics of Solids **4** (1956)
5. R. Hill, *Discontinuity relations in mechanics of solids*, Progress in Solid Mechanics, Vol. II, North-Holland, Amsterdam, 1961
6. A. S. Balchan and H. G. Drichamer, *High pressure electrical resistance cell and calibration points above* 100 *kilobars*, Rev. Sci. Instruments **32** (1961)
7. E. M. Shoemaker, *A Theory of linear plasticity for plane strain*, Arch. for Rational Mech. and Anal. **14** (1963)