NECESSARY CONDITIONS FOR SUBHARMONIC AND SUPERHARMONIC SYNCHRONIZATION IN WEAKLY NONLINEAR SYSTEMS*

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1. Introduction. The problem under consideration in this paper is that of a resonant system excited at an angular frequency, ω, not close to resonance. The system possesses weak nonlinearities which permit energy exchange between a response at the excitation frequency and a response close to resonance frequency. Under appropriate conditions these two responses can synchronize in a stable manner. Asymptotically then, the total response consists of one principal component at frequency ω and another at frequency $n\omega/m$, where n and m are positive integers. If $m/n < 1$ the resonance frequency is above the exciting frequency and the response is termed "superharmonic". For $m/n > 1$ the response is "subharmonic".

In a note [1] published in this journal, Lundquist investigated the occurrence of subharmonic oscillations in a passive nonlinear system with weak forcing. The analysis was restricted to the first order of approximation, that is, the response of the system at its resonance frequency was taken to be of order unity in magnitude and the conditions for the existence and stability of the oscillation were found by consideration of terms of order ϵ (where ϵ measures both the strength of the nonlinearity and the strength of the forcing function). Unfortunately, some of the terms considered were actually, in a subtle way, of higher order than ϵ. Had this been recognized the analysis would (correctly) have yielded only the null solutions. As it was, inclusion of the higher order terms gave (erroneously) subharmonic synchronization of a wide variety of orders independent of the kind of nonlinear function.

In the present paper we shall show that there is an intimate connection between the nonlinear function and the orders of subharmonics or superharmonics which may be synchronized. A procedure will be set for determining whether a certain nonlinearity might sustain a particular order of subharmonic or superharmonic and the special case of polynomial nonlinearity will be investigated in detail.

2. The Lundquist Problem. The differential equation considered by Lundquist, with a slight change in notation, is

$$\frac{d^2y}{dt^2} + y = \epsilon\left[F \sin(\omega t + \nu) - \beta \frac{dy}{dt} - f(y)\right], \tag{1}$$

with $f(y) = -f(-y)$; $\epsilon, \beta, F > 0$; $\epsilon \ll 1$.

He assumed the solution of (1) to be of the form

$$y = R \sin t + \epsilon\eta(t) \tag{2}$$

where $R = O(1)$ is the amplitude of the subharmonic. It is expected that periodic solutions

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of this type will be found with a period $2\pi + \tau_1$ where $\tau_1 = O(\epsilon)$. This period must in turn be a multiple $m/n(m, n$ are integers) of $2\pi/\omega$, and the order of the subharmonic solution, y, is n/m in conventional parlance. It follows then that $\omega - m/n = \tau_1\omega/2\pi$ or $\omega - m/n = \omega\, O(\epsilon)$. The relative phase between excitation and response for the equilibrium solution is established through ν.

A solution starting with initial conditions $y(0) = 0$, $dy/dt(0) = R$ will return to $y = 0$ with positive slope after the period $2\pi + \tau_1$. The discrepancies in the slope and relative phase at this point were determined following the procedure of Cartwright and Littlewood [2]. By imposing the conditions of periodicity for the solution, the amplitude R and phase ν were found to satisfy the equations:

$$\frac{\epsilon F}{\pi R(\omega^2 - 1)} \sin \omega\pi \sin (\omega\pi + \nu) = \frac{\epsilon\beta}{2\omega} + O(\epsilon^2) \tag{3}$$

$$\frac{\epsilon F}{\pi R(\omega^2 - 1)} \sin \omega\pi \cos (\omega\pi + \nu) = \frac{\epsilon}{2\pi R} \int_0^{2\pi} f(R \sin t) \sin t\, dt - \frac{\omega - m/n}{\omega} + O(\epsilon^2) \tag{4}$$

In deriving these equations it was assumed that a full number of force cycles are found in the period $2\pi + \tau_1$. This restricts n to be unity. The expressions (3) and (4) were then simplified to yield the amplitude-frequency characteristics of the subharmonics. According to this analysis, the order n/m of the subharmonic does not depend on the form of the nonlinearity. In fact, a more careful examination of the expressions (3) and (4) indicates that their left-hand sides are of $O(\epsilon^2)$ or higher in magnitude. There are two separate cases to be considered. First, the case of $\omega = O(1)$ (i.e., low order subharmonics): since $\omega - m/n = \omega O(\epsilon)$ then $\sin \omega\pi \simeq O(\epsilon)$ and the left-hand sides of (3) and (4) are of order of ϵ^2. On the other hand, if ω is of order $1/\epsilon$ in magnitude (i.e., higher order subharmonics are being examined) then $\sin \omega\pi \simeq O(1)$ and the coefficient $\epsilon F/\pi(\omega^2 - 1) = O(\epsilon^3)$, so again the left-hand sides of (3) and (4) may be neglected in comparison with terms of order ϵ in magnitude. In either case, the amplitude of the subharmonic is zero to the first order approximation and a higher order of approximation has to be carried out in order to detect it. This will be discussed in more detail in the following section.

The conclusions of Lundquist were checked by Hansson and Goransson [3] in an experiment with an electric circuit containing an iron core near saturation. Some of their observations were that the amplitude of the subharmonic was almost independent of the driving voltage and that the subharmonic appeared spontaneously in certain cases. These pecularities lead one to believe that the experiment was not a valid representation of the system discussed by Lundquist.

3. Formulation of the Problem and Method of Solution. Consider the system which is described by the differential equation

$$\frac{d^2x}{dt^2} + \epsilon f\left(x, \frac{dx}{dt}\right) + \omega_0^2 x = F \cos \omega t, \tag{5}$$

where ϵf represents the nonlinearities and possibly a small linear component as well. Physically, it is more interesting to consider strong subharmonic and superharmonic oscillations instead of weak ones (of order ϵ and higher in magnitude). For this reason, we shall focus our attention on the existence of synchronized oscillations of order unity in magnitude and consider strong forcing in (5). Next, anticipating a result to follow, we will permit the dissipation and the detuning to be of order ϵ.

Clearly, one part of the forced response of this system which is correct to order unity in magnitude is $S \cos \omega t$ where $S = F/(\omega^2 - \omega_0^2)$. This suggests the substitution:

$$x = y - S \cos \omega t. \tag{6}$$

Defining

$$\gamma = \left(\frac{n^2}{m^2}\omega^2 - \omega_0^2\right)\Big/\epsilon, \tag{7}$$

(5) becomes:

$$\frac{d^2y}{dt^2} + \frac{n^2}{m^2}\omega^2 y = -\epsilon\left[f\left(y - S\cos\omega t, \frac{dy}{dt} + S\omega\sin\omega t\right) - \gamma y\right]. \tag{8}$$

The solution of (8) may be approximated by a variety of techniques such as those by Poincaré [4], Van der Pol [5], Krylov–Bogoliubov–Mitropolski [6], [7] and Cole and Kevorkian [8]. We shall use the latter's technique and will represent the solution of (8) by an asymptotic expansion involving two time variables:

$$y(t, \tau, \epsilon) = y_0(t, \tau) + \epsilon y_1(t, \tau) + \epsilon^2 y_2(t, \tau) + \cdots \tag{9}$$

where $\tau = \epsilon t$ is the "slow" time, and the variable t and τ are treated as independent. In order to obtain uniformly valid second-, third-, and higher-order approximations of the solution, it becomes necessary to consider the additional terms in an expansion of the slow time scale

$$\tau = t(\epsilon + \epsilon^2 A_2 + \epsilon^3 A_3 + \cdots)$$

and further to expand the square of the natural frequency:

$$\omega_0^2 = \omega_0^2(1 + \epsilon B_1 + \epsilon^2 B_2 + \cdots).$$

The coefficients A_i and B_i are chosen appropriately so as to assure the boundedness of the solutions in the slow time variable. This will not be done here since we are only interested in the first order approximation to the solution. By substituting (9) into (8) and collecting terms of like order in ϵ, we obtain the equations:

$$\epsilon^0 \text{ terms:}\quad \frac{\partial^2 y_0}{\partial t^2} + \frac{n^2}{m^2}\omega^2 y_0 = 0, \tag{10}$$

$$\epsilon \text{ terms:}\quad \frac{\partial^2 y_1}{\partial t^2} + \frac{n^2}{m^2}\omega^2 y_1 = -f\left(y_0 - S\cos\omega t, \frac{\partial y_0}{\partial t} + S\omega\sin\omega t\right) + \gamma y_0 - 2\frac{\partial^2 y_0}{\partial t\,\partial\tau}. \tag{11}$$

The solution of (10) is of the form

$$y_0(t, \tau) = R(\tau)\cos\left[\frac{n}{m}\omega t + \varphi(\tau)\right], \tag{12}$$

where R and φ represent the amplitude and phase of the subharmonic $n/m < 1$ or superharmonic $n/m > 1$. The solution (12) is then substituted into the right-hand side of (11) and the secular terms are suppressed by:

$$2\frac{n}{m}\omega\frac{dR}{d\tau} = \left[f\left(y_0 - S\cos\omega t, \frac{\partial y_0}{\partial t} + S\omega\sin\omega t\right)\right]_{\sin((n/m)\omega t+\varphi)\text{ terms}^\dagger}, \tag{13}$$

$$2\frac{n}{m}\omega R\frac{d\varphi}{d\tau} = \left[f\left(y_0 - S\cos\omega t, \frac{\partial y_0}{\partial t} + S\omega\sin\omega t\right)\right]_{\cos((n/m)\omega t+\varphi)\text{ terms}^\dagger} - \gamma R. \tag{14}$$

A phase plane analysis may now be used to determine the behavior of R and φ. Elimination of the slow time, τ, from (13) and (14) gives:

$$\frac{dR}{R\,d\varphi} = \frac{[f]_{\sin((n/m)\omega t+\varphi)\text{ terms}}}{[f]_{\cos((n/m)\omega t+\varphi)\text{ terms}} - \gamma R}, \tag{15}$$

which will yield integral curves in the $R - \varphi$ phase. In general, (15) cannot be integrated in a closed form*; however, much may be learned from a study of the singular points of (15). These points are located at the common roots of the algebraic equations:

$$[f]_{\sin[(n/m)\omega t+\varphi]\text{ terms}} = 0, \tag{16}$$

$$[f]_{\cos[(n/m)\omega t+\varphi]\text{ terms}} = \gamma R. \tag{17}$$

The singular points represent all possible equilibrium values of the amplitude and phase and therefore include all synchronized solutions.

4. Some Necessary Conditions for Synchronizations. In a linear system there is no interaction between the response at the natural frequency y, and the forced response $S\cos\omega t$. Nonlinearities give interactions which may be classified as synchronous or asynchronous depending on whether they involve the relative phase of the two response components or not. Of special interest are those interactions which are themselves at the frequency of one or the other response components, and in particular at the natural frequency. Since the nonlinearity is assumed to be of order ϵ in the present analysis, the principal interaction components are of order ϵ and can be found by substituting the solution $y_0 - S\cos\omega t$ into the nonlinear function. For a particular polynomial nonlinearity, these principal components will generally be limited in number and there may or may not be a synchronous component at the natural frequency. If no such component of magnitude ϵ exists then it will be impossible to sustain a synchronized subharmonic or superharmonic under conditions requiring an interaction of this strength (e.g., dissipation in the system of order ϵ, or detuning of a self-excited oscillator by a frequency shift of order ϵ). For the purposes of the present work we are interested in systems where such a requirement applies, and can therefore state that the existence of a principal interaction component at the resonance frequency, which is synchronous, is necessary for subharmonic or superharmonic synchronization.

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†The notation

$$[f]_{\substack{\sin \\ \cos}\{((n/m)\omega t+\varphi)\}\text{ terms}} \quad \text{implies} \quad \frac{n\omega}{\pi m}\int_0^{2\pi m/n\omega}[f]\begin{Bmatrix}\sin\left(\dfrac{n}{m}\omega t + \varphi\right) \\ \cos\left(\dfrac{n}{m}\omega t + \varphi\right)\end{Bmatrix}dt,$$

i.e., the coefficients of the sin $[(n/m)\omega t + \varphi]$ or cos $[(n/m)\omega t + \varphi]$ terms of the Fourier expansion of the function f.

*Expression (15) can be integrated in the case of a conservative nonlinearity and this can be found in [9].

In terms of the formulation of the previous section, this condition may be interpreted as follows. If we denote that part of f which is nonlinear by f', then the interaction components at the natural frequency are

$$[f'(y - S\cos\omega t)]_{\left\{\begin{smallmatrix}\sin[(n/m)\omega t+\varphi]\\ \cos[(n/m)\omega t+\varphi]\end{smallmatrix}\right\}\text{ terms}}$$

and in order for these to be synchronous there must be an explicit dependence on the relative phase, φ. If there is no synchronous interaction (13) and (14) will have no singular points since each right-hand side will be independent of φ.

The action of the nonlinearity on either y or $S\cos\omega t$ generates harmonics of each component of magnitude ϵ. These harmonics along with the principal interaction components produce further interaction components of order ϵ^2 or higher. These new components widen the frequency range of the interaction and may produce subharmonic or superharmonic synchronization at a higher order of ϵ. Pursuing this course of investigation, Gambill and Hale [10] showed that if the nonlinearity was cubic, subharmonics of order $m/n = 2k + 1$ might be synchronized provided the system dissipation was of order ϵ^k. Their investigation is complimentary to the present one. It is not difficult to see how the two could be combined to give a general necessary existence criterion for any type of polynomial nonlinearity combined with any order of system dissipation but such an extension is beyond the scope of this study.

5. Relationship between Order of a Polynomial Nonlinearity and Frequency Range of Synchronized Oscillations. Consider a nonlinearity of the general polynomial form $f' = x^r(dx/dt)^s$. The $\sin[(n/m)\omega t + \varphi]$ and $\cos[(n/m)\omega t + \varphi]$ contributions of f' may be represented by the integrals:

$$\left.\begin{matrix} I_1'(R, S, \varphi) \\ I_2'(R, S, \varphi) \end{matrix}\right\} = \int_0^{2m\pi/n\omega} x^r\left(\frac{dx}{dt}\right)^s \left\{\begin{matrix} \sin\left(\frac{n}{m}\omega t + \varphi\right) \\ \cos\left(\frac{n}{m}\omega t + \varphi\right) \end{matrix}\right\} dt, \tag{18}$$

where

$$\begin{aligned} x &= R\cos\left(\frac{n}{m}\omega t + \varphi\right) - S\cos\omega t, \\ \frac{dx}{dt} &= -\frac{n}{m}\omega R\sin\frac{n}{m}\omega t + \varphi + S\omega\sin\omega t. \end{aligned} \tag{19}$$

Since it is cumbersome to evaluate (18) directly, we consider the generating function:

$$\left.\begin{matrix} I_1(\eta, R, S, \varphi) \\ I_2(\eta, R, S, \varphi) \end{matrix}\right\} = \int_0^{2m\pi/n\omega} \left(x + \eta\frac{dx}{dt}\right)^p \left\{\begin{matrix} \sin\left(\frac{n}{m}\omega t + \varphi\right) \\ \cos\left(\frac{n}{m}\omega t + \varphi\right) \end{matrix}\right\} dt, \tag{20}$$

where $p = r + s$; and then

$$I_{1,2}'(R, S, \varphi) = \frac{(p-s)!}{p!}\frac{d^sI_{1,2}}{d\eta^s}(0, R, S, \varphi). \tag{21}$$

By the use of (19) and trigonometric identities, the expression (20) may be simplified to give:

$$\left.\begin{matrix} I_1(\eta, R, S, \varphi) \\ I_2(\eta, R, S, \varphi) \end{matrix}\right\} = \int_0^{2m\pi/n\omega} \left[c \cos\left(\frac{n}{m}\omega t + \varphi + \zeta_{n/m}\right) - d \cos(\omega t + \zeta_1)\right]^p \cdot \left\{\begin{matrix} \sin\left(\frac{n}{m}\omega t + \varphi\right) \\ \cos\left(\frac{n}{m}\omega t + \varphi\right) \end{matrix}\right\} dt, \tag{22}$$

where

$$c = R\left(1 + \eta^2 \frac{n^2}{m^2}\omega^2\right)^{1/2}; \qquad \zeta_{n/m} = \tan^{-1} \eta \frac{n}{m}\omega;$$

$$d = S(1 + \eta^2\omega^2)^{1/2}; \qquad \zeta_1 = \tan^{-1} \eta\omega.$$

The contributions at the resonance frequency for the nonlinearity $x^n(dx/dt)^s$ are now found according to the operations in (21). From the discussion in the previous section, I_1' or I_2' must exhibit an explicit dependence on φ for synchronization to occur. This is equivalent to requiring an explicit dependence of I_1 or I_2 on φ.

Consider I_1' and I_2' for the special nonlinear function $f' = x^p$:

$$\left.\begin{matrix} I_1'(R, S, \varphi) \\ I_2'(R, S, \varphi) \end{matrix}\right\} = \int_0^{2m\pi/n\frac{1}{2}} \left[R \cos\left(\frac{n}{m}\omega t + \varphi\right) - S \cos \omega t\right] \left\{\begin{matrix} \sin\left(\frac{n}{m}\omega t + \varphi\right) \\ \cos\left(\frac{n}{m}\omega t + \varphi\right) \end{matrix}\right\} dt \tag{23}$$

Comparing this with (22), we note that except for the substitution of c and d for R and S for a redefinition of the relative phase to $\varphi + \zeta_{n/m} - \zeta_1$ the two functional forms are identical. Thus, if I_1' or I_2' in (23) exhibit dependence on φ, I_1 or I_2 in (22) will do so also. This is to say that the test for possible existence of synchronized solutions can be performed with x^p in place of $x^r(dx/dt)^s$.

Expanding x^p in a binomial series, we obtain:

$$\left[R \cos\theta - S \cos\frac{m}{n}(\theta - \varphi)\right]^p = \sum_{k=0}^{p} \frac{(-1)^k p!}{(p-k)!\,k!} R^{p-k} S^k \cos^{p-k}\theta \cos^k \frac{m}{n}(\theta - \varphi), \tag{24}$$

where $\theta = (n/m)\omega t + \varphi$. Since what really matters is the range of the θ frequencies in the summation and not the values of the coefficients, only the arguments of the trigonometric terms need be retained. In the case of odd p, the following arguments are produced:
for k odd:

$$\left[(p - k - 2j) \pm \frac{m}{n}(k - 2i)\right]\theta \mp \frac{m}{n}\varphi(k - 2i), \tag{25}$$

where

$$1 \le k \le p, \qquad 0 \le j \le \frac{p-k}{2}, \qquad 0 \le i \le \frac{k-1}{2},$$

and for k even:

$$\left[(p - k - 2i') \pm \frac{m}{n}(k - 2j')\right]\theta \mp \frac{m}{n}\varphi(k - 2j'), \tag{26}$$

where

$$2 \le k \le p - 1, \qquad 0 \le j' \le \frac{k-2}{2}, \qquad 0 \le i' \le \frac{p-k-1}{2}.$$

The nonlinearity will contribute cos θ and sin θ terms if:

$$(p - k - 2j) \pm \frac{m}{n}(k - 2i) = \pm 1; \qquad (p - k - 2i') \pm \frac{m}{n}(k - 2j') = \pm 1. \tag{27}$$

Since $(p - k - 2j)$ min $= 0$, $(k - 2i)$ min $= 1$; $(p - k - 2i')$ min $= 1$, $(k - 2j')$ min $= 2$ then for $m/n \neq 1$, the positive sign on the left-hand sides of (27) may be dropped. Similarly, the positive sign on the right-hand sides of (27) may be excluded since the range of variation of j includes the effect of the positive sign. Therefore,

$$\frac{m}{n} = \frac{p + 1 - k' - 2v}{k'}; \qquad p = \text{odd}, \tag{28}$$

where

$$k' = k - 2i = 1, 2, 3, \cdots, p$$

$$v = 0, 1, 2, 3, \cdots \begin{cases} \dfrac{p - k'}{2} & \text{for } k' \text{ odd} \\ \dfrac{p - k' - 1}{2} & \text{for } k' \text{ even} \end{cases}.$$

The same procedure in the case of an even order nonlinearity yields:

$$\frac{m}{n} = \frac{p + 1 - k' - 2v'}{k'}; \qquad p = \text{even}, \tag{29}$$

where

$$k' = 1, 2, 3, \cdots, p$$

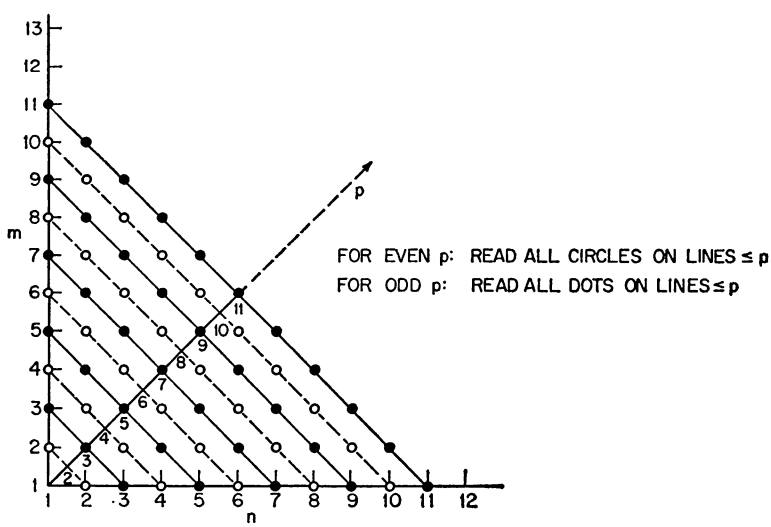

FIG. 1. Orders of Sub- and Super-Harmonics m/n for a Nonlinearity of Total Order p.

$$v' = 0, 1, 2, 3, \cdots \begin{cases} \dfrac{p - k' - 1}{2} & \text{for } k' \text{ odd} \\ \dfrac{p - k'}{2} & \text{for } k' \text{ even} \end{cases}.$$

A plot of m versus n for various values of p is given in Figure 1. It is interesting to observe the symmetry between the subharmonics and superharmonics that can be sustained with a polynomial nonlinearity of a given power. The order of the subharmonic is merely an inversion of the order of the superharmonic. This may be interpreted to indicate that the interaction of y and $S \cos \omega t$ which produces a synchronous component at the y-frequency will also produce one at the S-frequency (ω). In fact, for x^p which is a conservative function an equal and opposite energy exchange must exist. Even for a nonconservative function ($x^{p-1}\, dx/dt$, for example), a reciprocal exchange exists. This is treated in detail in [9]. It is further observed from Figure 1 that the order of the synchronized oscillations may be fractional (e.g., $m/n = \frac{3}{2}, \frac{2}{3}$ for $p = 4$). Finally, odd orders exist only when p is odd, while even orders exist when p is odd or even.

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