ON STRESS FUNCTIONS IN CLASSICAL ELASTICITY*

BY

MARVIN STIPPES

University of Illinois

1. Introduction. Suppose a material body occupies a bounded simply-connected region D' of space. Its interior and boundary are denoted by D and ∂D respectively. If the material is linear, homogeneous, isotropic and elastic, then the displacement vector $\mathbf{u}$ and the stress dyadic $\mathbf{S}$, in the absence of body force, satisfy the following equations

$$
\begin{gathered}
\nabla \cdot \mathbf{S} = 0 \text{ in } D \\
\mathbf{S} = \lambda(\nabla \cdot \mathbf{u})\mathbf{I} + 2\mu\mathbf{E}, \qquad 2\mathbf{E} = \nabla\mathbf{u} + \mathbf{u}\nabla \text{ in } D'
\end{gathered}
\tag{1.1}
$$

where ∇ is the gradient operation, λ, μ are Lame's constants, $\mathbf{I}$ the idemfactor and $\mathbf{E}$ the strain dyadic. The notation for vectors, dyadics and the various products is that of [1][1].

Two distinct formulations of the boundary value problems occur: one, by means of Cauchy's equations, which take the displacement components as dependent variables; and the other, the Beltrami–Michell system, which employs the stresses in a parallel role. The differential equations corresponding to the former are

$$
(\lambda + \mu)\nabla\nabla \cdot \mathbf{u} + \mu\nabla^2\mathbf{u} = 0,
\tag{1.2}
$$

and the latter become

$$
\nabla \cdot \mathbf{S} = 0, \qquad (1 + \nu)\nabla^2\mathbf{S} + \nabla p \nabla = 0, \qquad p = \mathbf{S} : \mathbf{I}
\tag{1.3}
$$

where ν is Poisson's ratio. (See, for example, [2] p. 73-75.)

A "general solution" of either (1.2) or (1.3) is a representation of $\mathbf{u}$ or $\mathbf{S}$ with sufficient functional arbitrariness to span the class of all $\mathbf{u}$ or $\mathbf{S}$, C^2 in D, that satisfy (1.2) or (1.3) there. Such forms can be obtained through the use of linear differential transformations of $\mathbf{u}$ or $\mathbf{S}$ to a new system of dependent variables satisfying simpler differential equations. Two of the most useful general solutions of (1.2), obtained by this technique, are Papkovich's[2]

$$
\mu\mathbf{u} = \boldsymbol{\psi} - \frac{1}{4(1 - \nu)} \nabla(\phi + \mathbf{r} \cdot \boldsymbol{\psi}); \qquad \nabla^2\phi = 0, \qquad \nabla^2\boldsymbol{\psi} = 0,
\tag{1.4}
$$

and Galerkin's[3]

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[1]Numbers in square brackets refer to bibliographical entries.

[2]As is well-known, V. J. Boussinesq offered elements of Eq. (1.4). An early appearance of this form with $\phi = 0$ or $\mathbf{k} \cdot \boldsymbol{\psi} = 0$ is in a series of papers by E. Fontaneau [3], [4].

[3]Although (1.5) is popularly attributed to B. G. Galerkin [5] and C. Somigliana [6], [7], V. J. Boussinesq presented the basic structure of this solution in his classic treatise [8], p. 284 as noted in [18], [19], and [17]. Having discovered it in 1883, he predates either of these men.

$$\mu \mathbf{u} = \nabla^2 \mathbf{g} - \frac{1}{2(1-\nu)} \nabla\nabla \cdot \mathbf{g}; \qquad \nabla^4 \mathbf{g} = 0. \tag{1.5}$$

The point of departure in applying this method to (1.3) is an expression for solenoidal $\mathbf{S}$. In this way (1.3a) is automatically satisfied, and it only remains to form a compatible state of stress. E. Beltrami [9] gave

$$\mathbf{S} = \nabla \times \mathbf{T} \times \nabla, \qquad \mathbf{T} = \mathbf{T}_c \tag{1.6}$$

as a general solution of (1.3a) tacitly assuming completeness for $\mathbf{T}$ C^3 in D. Although his result is over seventy years old, it has only been recently observed [10] that if ∂D^* is any closed surface in D', $\mathbf{n}$ the outer normal to ∂D^* and $\mathbf{r}$ the position vector, then

$$\int_{\partial D^*} (\mathbf{n} \cdot \mathbf{S})\, dQ = 0, \qquad \int_{\partial D^*} \mathbf{r} \times (\mathbf{n} \cdot \mathbf{S})\, dQ = 0 \tag{1.7}$$

for any $\mathbf{S}$ of the form of (1.6). Since stress fields satisfying (1.3a) exist which violate (1.7) for arbitrary ∂D^*, Beltrami's solution cannot contain all equilibrated states when $\mathbf{T}$ is too smooth. However, when $\mathbf{S}$ is totally self-equilibrated, that is, (1.7) holds, and if ∂D is C^3 and $\mathbf{S}$ is C^3 in D, C^2 in D', M. E. Gurtin [11] has established that Beltrami's solution is complete. Furthermore, he has shown that if $\mathbf{S}$ is C^3 in D (which can be multiply-connected), then every equilibrated stress state is of the form

$$\mathbf{S} = \nabla \times \mathbf{T} \times \nabla + \nabla^2(\nabla \mathbf{g} + \mathbf{g}\nabla) - \nabla(\nabla \cdot \mathbf{g})\nabla \tag{1.8}$$

provided $\mathbf{T}$ ranges over the class of symmetric dyadics C^3 in D and $\mathbf{g}$ over the set of regular biharmonic vectors in D. He subsequently reduced this to

$$\mathbf{S} = \nabla \times \mathbf{T} \times \nabla + \nabla \boldsymbol{\psi} + \boldsymbol{\psi} \nabla - \mathbf{I} \nabla \cdot \boldsymbol{\psi} \tag{1.9}$$

where $\boldsymbol{\psi}$ extends over all vectors, regular and harmonic in D.

Our purpose is to form solutions of (1.3a) as the Euler equations of certain variational theorems, and then to adjust $\mathbf{T}$ and $\mathbf{g}$ or $\boldsymbol{\psi}$ so that $\mathbf{S}$ given by (1.8) or (1.9) is compatible. When this is done, we find $\mathbf{g}$ in (1.8) is Galerkin's vector and $\boldsymbol{\psi}$ in (1.9) to be the Papkovich vector potential.

Variational Theorems. Consider the integral

$$V = \int_D \mathbf{S} : \mathbf{E}\, dQ - \int_{\partial D} \mathbf{t} \cdot \mathbf{u}\, dQ \tag{2.1}$$

where $\mathbf{S}$ and $\mathbf{t}$ are presumed known, $\mathbf{t} = \mathbf{n} \cdot \mathbf{S}$, and $\mathbf{E}$ is the symmetric part of $\nabla \mathbf{u}$. In what follows all variables are subjected to continuity requirements sufficient to gaurantee use of the divergence theorem.

Theorem 1: If $\mathbf{u}$, $\mathbf{E}$ are subject to variations $\delta \mathbf{u}$, $\delta \mathbf{E}$ which satisfy (1.1c), then the Euler equations necessary for stationary values of V are

$$\mathbf{S} = -\mathbf{U}, \qquad \nabla \cdot \mathbf{U} = 0 \tag{2.2}$$

where $\mathbf{U}$ is the symmetric Lagrange multiplier for the side conditions given by (1.1c).

The result follows by writing

$$W = V + (\tfrac{1}{2}) \int_D \mathbf{U} : (2\mathbf{E} - \nabla \mathbf{u} - \mathbf{u}\nabla)\, dQ \tag{2.3}$$

from which

$$\delta W = \int_D [(\mathbf{S} + \mathbf{U}) : \delta\mathbf{E} + \delta\mathbf{u}\cdot(\nabla\cdot\mathbf{U})]\, dQ + \text{surface integral}. \tag{2.4}$$

On setting the coefficients of $\delta\mathbf{u}$, $\delta\mathbf{E}$ equal to zero, we have Theorem 1. This is the counterpart of L. Donati's theorem [12] in which $\mathbf{S}$, instead, is varied subject to the side conditions (1.3a) in D and $\mathbf{n}\cdot\delta\mathbf{S} = 0$ on ∂D. In this case the multiplier is a vector, say $\boldsymbol{\lambda}$, such that $2\mathbf{E} = \nabla\boldsymbol{\lambda} + \boldsymbol{\lambda}\nabla$ is necessary for V to be stationary.

Suppose now that $\boldsymbol{\omega}$ is the infinitesimal rotation vector. Then the antisymmetric part of $\nabla\mathbf{u}$ is $-\mathbf{I}\times\boldsymbol{\omega}$. We can express V of (2.1) as a functional of $\mathbf{E}$, $\boldsymbol{\omega}$ by the following device. Since $\mathbf{t}$ is presumed known on ∂D and if it meets (1.7a) in which ∂D^* is replaced by ∂D, then there exists a vector $\boldsymbol{\chi}$ such that

$$\nabla^2\boldsymbol{\chi} = 0 \text{ in } D, \qquad \partial\boldsymbol{\chi}/\partial n \equiv \mathbf{n}\cdot\nabla\boldsymbol{\chi} = \mathbf{t} \quad \text{on} \quad \partial D. \tag{2.5}$$

Consequently, (2.1) can be rewritten as

$$V^* = \int_D [\mathbf{S} : \mathbf{E} - \tfrac{1}{2}(\nabla\boldsymbol{\chi} + \boldsymbol{\chi}\nabla) : \mathbf{E} + \nabla\boldsymbol{\chi} : (\mathbf{I}\times\boldsymbol{\omega})]\, dQ \tag{2.6}$$

where now $\mathbf{E}$, $\boldsymbol{\omega}$ must satisfy

$$\boldsymbol{\omega}\nabla - \nabla\times\mathbf{E} = 0. \tag{2.7}$$

Relative to (2.6) we can formulate the following theorem.

Theorem 2. If $\mathbf{E}$, $\boldsymbol{\omega}$ are subject to variations $\delta\mathbf{E}$, $\delta\boldsymbol{\omega}$ consistent with (2.7), then the Euler equations necessary for stationary values of V^* are (1.9).

To establish Theorem 2, let $\mathbf{G}$ be the Lagrange multiplier with which (2.7) is incorporated into (2.6). Then

$$W^* = V^* + \int_D \mathbf{G} : (\boldsymbol{\omega}\nabla - \nabla\times\mathbf{E})\, dQ \tag{2.8}$$

Computing δW^*

$$\begin{aligned}\delta W^* = \int_D ([\mathbf{S} - \tfrac{1}{2}(\nabla\boldsymbol{\chi} + \boldsymbol{\chi}\nabla) - \tfrac{1}{2}(\nabla\times\mathbf{G} - \mathbf{G}_c\times\nabla)] : \delta\mathbf{E}\\ - \delta\boldsymbol{\omega}\cdot[\nabla\cdot(\mathbf{G}_c + \mathbf{I}\times\boldsymbol{\chi})])\, dQ + \text{surface integral}\end{aligned} \tag{2.9}$$

where $\mathbf{G}_c$ is the conjugate of $\mathbf{G}$. The corresponding Euler equations are

$$\mathbf{S} = \tfrac{1}{2}(\nabla\boldsymbol{\chi} + \boldsymbol{\chi}\nabla) + \tfrac{1}{2}(\nabla\times\mathbf{G} - \mathbf{G}_c\times\nabla), \qquad \nabla\cdot(\mathbf{G}_c + \mathbf{I}\times\boldsymbol{\chi}) = 0. \tag{2.10}$$

To place the first of these in Gurtin's form, recall that according to the Stokes–Helmholtz decomposition theorem, one can always write

$$\mathbf{G}_c + \mathbf{I}\times\boldsymbol{\chi} = \nabla\boldsymbol{\vartheta} + \nabla\times\mathbf{U}, \qquad \nabla^2\boldsymbol{\vartheta} = 0. \tag{2.11}$$

The harmonic character of $\boldsymbol{\vartheta}$ is a consequence of (2.10b). Inserting $\mathbf{G}$ and $\mathbf{G}_c$ from (2.11) into (2.10a) and reducing, we find

$$\begin{aligned}\mathbf{S} = -\nabla\times[\tfrac{1}{2}(\mathbf{U} + \mathbf{U}_c)]\times\nabla + \nabla(\boldsymbol{\chi} + \tfrac{1}{2}\nabla\times\boldsymbol{\vartheta})\\ + (\boldsymbol{\chi} + \tfrac{1}{2}\nabla\times\boldsymbol{\vartheta})\nabla - \mathbf{I}\nabla\cdot(\boldsymbol{\chi} + \tfrac{1}{2}\nabla\times\boldsymbol{\vartheta}).\end{aligned} \tag{2.12}$$

Setting

$$\mathbf{T} = -\tfrac{1}{2}(\mathbf{U} + \mathbf{U}_c), \qquad \boldsymbol{\psi} = \boldsymbol{\chi} + \tfrac{1}{2}\nabla \times \boldsymbol{\vartheta},$$

and noting that $\mathbf{T}$ is self-conjugate and $\boldsymbol{\psi}$ is harmonic, we have (1.9).[4]

Gurtin's original form for solenoidal $\mathbf{S}$, as embodied in (1.8), can be obtained in a similar manner by computing, instead of $\boldsymbol{\chi}$ as given by (2.5), a biharmonic vector $\boldsymbol{\phi}$ whose Laplacian is equal to $\mathbf{t}$ on ∂D. Now (2.12) becomes

$$\mathbf{S} = -\nabla \times [\tfrac{1}{2}(\mathbf{U} + \mathbf{U}_c)] \times \nabla + \nabla[\nabla^2\boldsymbol{\phi} + \tfrac{1}{2}\nabla \times \boldsymbol{\vartheta}] + [\nabla^2\boldsymbol{\phi} + \tfrac{1}{2}\nabla \times \boldsymbol{\vartheta}]\nabla - \mathbf{I}\nabla\cdot(\nabla^2\boldsymbol{\phi} + \tfrac{1}{2}\nabla \times \boldsymbol{\vartheta}). \tag{2.13}$$

If $\nabla^2\mathbf{a} = \frac{1}{2}\nabla \times \boldsymbol{\vartheta}$, then $\mathbf{a}$ is biharmonic in D and (2.13) can be written as

$$\mathbf{S} = -\nabla \times [\tfrac{1}{2}(\mathbf{U} + \mathbf{U}_c)] \times \nabla + \nabla^2[\nabla(\boldsymbol{\phi} + \mathbf{a}) + (\boldsymbol{\phi} + \mathbf{a})\nabla] - \mathbf{I}\nabla^2\nabla\cdot(\boldsymbol{\phi} + \mathbf{a}). \tag{2.14}$$

Since

$$\nabla \times \beta\mathbf{I} \times \nabla = \nabla\beta\nabla - \nabla^2\beta\mathbf{I}, \tag{2.15}$$

we have

$$\mathbf{S} = \nabla \times [-\tfrac{1}{2}(\mathbf{U} + \mathbf{U}_c) + \mathbf{I}\nabla\cdot(\boldsymbol{\phi} + \mathbf{a})] \times \nabla + \nabla^2[\nabla(\boldsymbol{\phi} + \mathbf{a}) + (\boldsymbol{\phi} + \mathbf{a})\nabla] - \nabla[\nabla\cdot(\boldsymbol{\phi} + \mathbf{a})]\nabla \tag{2.16}$$

for (2.14). Setting

$$\mathbf{T} = -\tfrac{1}{2}(\mathbf{U} + \mathbf{U}_c) + \mathbf{I}\nabla\cdot(\boldsymbol{\phi} + \mathbf{a}), \qquad \mathbf{g} = \boldsymbol{\phi} + \mathbf{a}$$

we note that $\mathbf{T}$ is symmetric and $\mathbf{g}$ is biharmonic and (2.16) is Gurtin's original form as given by (1.8).

When (2.7) is replaced by the compatibility conditions

$$\nabla \times \mathbf{E} \times \nabla = 0 \tag{2.17}$$

and the symmetric multiplier is $-\mathbf{T}$, the equation corresponding to (2.9) is

$$\delta W^{**} = \int_D (\mathbf{S} - \nabla \times \mathbf{T} \times \nabla - \nabla\boldsymbol{\psi} - \boldsymbol{\psi}\nabla + \mathbf{I}\nabla\cdot\boldsymbol{\psi}) : \delta\mathbf{E}\, dQ + \text{surface integral} \tag{2.18}$$

where $\boldsymbol{\chi}$ has been replaced by $2\boldsymbol{\psi}$. Now the Euler equations are obtained directly in the form of (1.9). Again (1.8) results by the same observation which led to the formation

[4]The derivation of (2.12) might lead one to consider (i): $\mathbf{S} = \nabla \times \mathbf{T} \times \nabla + \nabla(\nabla \times \boldsymbol{\vartheta}) + (\nabla \times \boldsymbol{\vartheta})\nabla$. $\nabla^2\boldsymbol{\vartheta} = 0$ as an alternate general solution of Eq. (1.3a). By the Stokes-Helmholtz theorem, one could write (ii): $\boldsymbol{\psi} = \nabla p + \nabla \times \mathbf{q}$ in Eq. (1.9). If, for example, $\boldsymbol{\psi} = \mathbf{k}\rho^{-1}$ in the spherical shell region $a < \rho < b$, there are no regular harmonic functions p, $\mathbf{q}$ in this same region for which (ii) holds and so (i) cannot be complete when $\boldsymbol{\vartheta}$ ranges over the class of regular harmonic functions.

of (2.13) in terms of a biharmonic vector. C. Truesdell [13] derived (1.6) using $-\mathbf{T}$ as the symmetric multiplier for (2.17) in conjunction with (2.1).[5]

We see then that beginning with the functional of (2.1), we obtain successively stronger statements concerning the form of the Euler equations as the side conditions proceed from the kinematical definition of (1.1) through the intermediary of the infinitesimal rotation as contained in (2.7) and finally to an introduction of the compatibility conditions.

Compatible S. With the general solution of (1.3) known, it is only necessary to adjust $\mathbf{T}$ and $\mathbf{g}$ or $\boldsymbol{\psi}$ so that (1.3b) is satisfied. G. Morera [14] presented a solution of this equation basing $\mathbf{S}$ on a form of (1.6) in which $\mathbf{T}$ consists of diagonal elements in a cartesian coordinate system (Maxwell's stress functions). V. Blokh [15] established that if Beltrami's dyadic is of the form

$$\mathbf{T} = (1 - \nu)\nabla^2\mathbf{B} + \mathbf{I}(\nabla\cdot\mathbf{B}\cdot\nabla), \qquad \nabla^4\mathbf{B} = 0 \tag{3.1}$$

then $\mathbf{S}$ as given by (1.6) is compatible. It is clear that Blokh's results are limited in use to totally self-equilibrated systems whenever $\mathbf{B}$ is required to be a regular biharmonic dyadic.

A solution of (1.3b) employing (1.9) is obtained by adding to $\mathbf{T}$ of (3.1), a particular integral $\mathbf{T}_0$ of

$$\nabla \times ((1 + \nu)\nabla^2\mathbf{T} + [\nabla\cdot\mathbf{T}\cdot\nabla - \nabla^2(\mathbf{T} : \mathbf{I})]\mathbf{I} - (\nabla\cdot\boldsymbol{\psi})\mathbf{I}) \times \nabla = 0, \tag{3.2}$$

which is taken in the form

$$\mathbf{T}_0 = a\mathbf{I}. \tag{3.3}$$

It is readily verified that a can be made to satisfy

$$-(1 - \nu)\nabla^2 a = \nabla\cdot\boldsymbol{\psi}. \tag{3.4}$$

In solving (3.4) we may choose a from one or the other of two possibilities:

$$-\frac{\mathbf{r}\cdot\boldsymbol{\psi}}{2(1 - \nu)}, \qquad \frac{1}{4\pi(1 - \nu)}\int_D \frac{\nabla\cdot\boldsymbol{\psi}}{R}\,dQ, \tag{3.5}$$

where the integral is a Newtonian potential. With $\mathbf{T}$ given by (3.1), (1.9) becomes

$$\mathbf{S} = \nabla \times \left[\mathbf{T} - \frac{\mathbf{r}\cdot\boldsymbol{\psi}}{2(1 - \nu)}\mathbf{I}\right] \times \nabla + \nabla\boldsymbol{\psi} + \boldsymbol{\psi}\nabla - \mathbf{I}\nabla\cdot\boldsymbol{\psi}, \tag{3.6}$$

or

$$S = \nabla \times \left[\mathbf{T} + \frac{\mathbf{I}}{4\pi(1 - \nu)}\int_D \frac{\nabla\cdot\boldsymbol{\psi}}{R}\,dQ\right] \times \nabla + \nabla\boldsymbol{\psi} + \boldsymbol{\psi}\nabla - \mathbf{I}\nabla\cdot\boldsymbol{\psi}. \tag{3.7}$$

If, on the other hand, $\mathbf{S}$ is expressed by (1.8), an analysis similar to that for (1.9) yields

[5] He further argued his solution was complete for arbitrary regions and smooth enough equilibrated $\mathbf{S}$. Since the divergence theorem was employed in the analysis, the manner in which the multiplier was used assumed it to be at least C^2 in D. This is consistent with Gurtin's observations about (1.6). For if D' is periphractic with but a single cavity, it can be divided into two non-periphractic regions D^*, D^{**} by introducting a suitable plane cut. Now a $\mathbf{T}^*$, $\mathbf{T}^{**}$ always exist in D^*, D^{**} such that (1.6) holds in each region. If $\mathbf{S}$ is not totally self-equilibrated in D', $\mathbf{T}^*$, $\mathbf{T}^{**}$ must be discontinuous across the cut and so Truesdell's derivation is not applicable.

$$\mathbf{S} = \nabla \times \left[\mathbf{T} - \frac{\nabla \cdot \mathbf{g}}{(1-\nu)} I\right] \times \nabla + \nabla^2(\nabla\mathbf{g} + \mathbf{g}\nabla) - \nabla(\nabla\cdot\mathbf{g})\nabla. \tag{3.8}$$

Writing Hooke's law as

$$\nabla(\mu\mathbf{u}) + (\mu\mathbf{u})\nabla = \mathbf{S} - \frac{\nu}{(1+\nu)}(\mathbf{S} : \mathbf{I})\mathbf{I}, \tag{3.9}$$

we find that the three previous forms for $\mathbf{S}$ in conjunction with (3.1) reduce (3.9) to an equation of the type

$$\nabla(\mu\mathbf{u} - \mathbf{v}) + (\mu\mathbf{u} - \mathbf{v})\nabla = 0 \tag{3.10}$$

from which it follows that $\mu\mathbf{u} - \mathbf{v} = \boldsymbol{\omega}_0 \times \mathbf{r} + \mathbf{a}$ where $\boldsymbol{\omega}_0$, $\mathbf{a}$ are constant vectors. The nature of $\mathbf{v}$ in each case allows us to absorb $\boldsymbol{\omega}_0 \times \mathbf{r} + \mathbf{a}$ into it and as a result, the displacement fields corresponding to (3.6), (3.7) and (3.8) are

$$\mu\mathbf{u} = \boldsymbol{\psi} - \frac{\nabla(\mathbf{r}\cdot\boldsymbol{\psi})}{4(1-\nu)} + \nabla\cdot[\tfrac{1}{2}(\mathbf{T} : \mathbf{I})\mathbf{I} - \mathbf{T}], \tag{3.11}$$

$$\mu\mathbf{u} = \boldsymbol{\psi} + \frac{\nabla}{2(1-\nu)}\left(\frac{1}{4\pi}\int_D \frac{\nabla\cdot\boldsymbol{\psi}}{R}\,dQ\right) + \nabla\cdot[\tfrac{1}{2}(\mathbf{T} : \mathbf{I})\mathbf{I} - \mathbf{T}], \tag{3.12}$$

$$\mu\mathbf{u} = \nabla^2\mathbf{g} - \frac{1}{2(1-\nu)}\nabla\nabla\cdot\mathbf{g} + \nabla\cdot[\tfrac{1}{2}(\mathbf{T} : \mathbf{I})\mathbf{I} - \mathbf{T}]. \tag{3.13}$$

When $\mathbf{T}$ in (3.11) is computed from (3.1) in which

$$\mathbf{B} = \frac{\mathbf{I}}{8\pi(1-\nu)(2-\nu)}\int_D \frac{\phi}{R}\,dQ, \qquad \nabla^2\phi = 0,$$

then (3.11) reduces to (1.4) and so $\boldsymbol{\psi}$ in this form is Papkovich's vector potential.

The displacement fields in (3.12) and (3.13) are complete with $\mathbf{T} = 0$ since under these conditions, the first of these is the Naghdi–Hsu solution [16] and the second Galerkin's solution (1.5).

In conclusion, we observe that H. Schaefer [17] had expressed solutions of (1.3a) in the form

$$\mathbf{S} = -\nabla^2\mathbf{T} + \nabla(\nabla\cdot\mathbf{T}) + (\nabla\cdot\mathbf{T})\nabla - (\nabla\cdot\mathbf{T}\cdot\nabla)\mathbf{I}, \qquad \mathbf{T} = \mathbf{T}_c. \tag{3.14}$$

For $\mathbf{S}$ to be compatible, he finds it sufficient to take

$$\mathbf{T} = \boldsymbol{\Theta} - (\tfrac{1}{2}\Omega)\mathbf{I}, \qquad \boldsymbol{\Theta} = \boldsymbol{\Theta}_c, \qquad \nabla^2\boldsymbol{\Theta} = 0, \qquad (1-\nu)\nabla^2\Omega = \nabla\cdot\boldsymbol{\Theta}\cdot\nabla, \tag{3.15}$$

thus obtaining

$$\begin{aligned} \mathbf{S} &= (\nabla^2\Omega)\mathbf{I} - \nabla\Omega\nabla + \nabla\boldsymbol{\vartheta} + \boldsymbol{\vartheta}\nabla - (\nabla\cdot\boldsymbol{\vartheta})\mathbf{I}, \\ \boldsymbol{\vartheta} &= \nabla\cdot\boldsymbol{\Theta}, \end{aligned} \tag{3.16}$$

for the stress field. The corresponding displacements are

$$\mu\mathbf{u} = \boldsymbol{\vartheta} - \tfrac{1}{2}\nabla\Omega, \tag{3.17}$$

and he concludes this is of the same character as Eq. (1.4) or (1.5). Because of the known identity

$$\nabla \times [(\mathbf{P} : \mathbf{I})\mathbf{I} - \mathbf{P}] \times \nabla = \nabla(\nabla\cdot\mathbf{P}) + (\nabla\cdot\mathbf{P})\nabla - \mathbf{I}(\nabla\cdot\mathbf{P}\cdot\nabla) - \nabla^2\mathbf{P},$$

(3.14) is nothing more than an alternate form of Beltrami's solution, (1.6), and is, therefore, subject to the same restrictions. If we write $\nabla \cdot \mathbf{\Theta}$ for $\boldsymbol{\vartheta}$ in (3.16) and set

$$\mathbf{\Theta} = \tfrac{1}{2}(1 - \nu)\mathbf{I}\nabla^2(\mathbf{B} : \mathbf{I}) - (1 - \nu)\nabla^2\mathbf{B}, \qquad \Omega = -\nabla\cdot\mathbf{B}\cdot\nabla,$$

then **S** becomes

$$S = \nabla \times [(1 - \nu)\nabla^2\mathbf{B} + \mathbf{I}(\nabla\cdot\mathbf{B}\cdot\nabla)] \times \nabla.$$

Strictly speaking, $\boldsymbol{\vartheta}$ in (3.15) must be the divergence of a symmetric harmonic dyadic. If, for example, D' is the spherical shell region $a \leq \rho \leq b$ and $\boldsymbol{\vartheta} = \mathbf{k}\rho^{-1}$, then $\mathbf{\Theta}$ cannot be regular in D. In view of Gurtin's work, it is fortuitous that **S** in (3.15) and therefore **u** in (3.17) could be made complete by a suitable choice of Ω consistent with (3.15), and regularity requirements. The role of regularity in completeness cannot be overemphasized. Indeed, it is probably so that any representation employing sufficiently smooth functions and complete in a special kind of domain (simply-connected, star-shaped, non-periphractic, etc.) remains complete in any reasonable type of region provided one relaxes the regularity conditions on the functions used in the representation. For example, footnote 5 bears this out for the case of (1.6).

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[6]The solution presented in this paper was rediscovered by M. Stippes and H. L. Langhaar and published in the Journal of the Franklin Institute in 1954.