NOTE ON THE GENERAL SOLUTION OF THE HEAT EQUATION*

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In a recent issue of this journal Winer [1] presented a solution to the heat equation

$$\left(\frac{1}{\kappa}\frac{\partial}{\partial t} - \nabla^2\right)T(\mathbf{r}, t) = \frac{Q(\mathbf{r}, t)}{K}, \qquad \mathbf{r} \text{ in } R, \quad t > 0 \tag{1}$$

in a stationary, homogeneous, isotropic, finite region R with constant thermal properties. In equation (1) $K > 0$ denotes thermal conductivity; $\kappa > 0$, thermal diffusivity; $Q(\mathbf{r}, t)$, a prescribed volume heat source per unit time and per unit volume; $\mathbf{r}$, position of a point in R; $T(\mathbf{r}, t)$, unsteady temperature field in R; t, time; and ∇^2 the Laplacian. Winer's solution is based on the assumptions that $Q(\mathbf{r}, t)$ is separable in $\mathbf{r}$ and t, that the initial temperature field is uniform throughout R, and that the boundaries of R are maintained at this constant initial temperature. Moreover, except when the separable source Q is a Dirac delta function in t, the solution given in [1] is not very suitable for numerical computations since the result of evaluation of the time integral appearing in Eq. (2) of [1] is to introduce slowly converging series expressions.

The purpose of this note is to summarize the general solutions of Eq. (1) not subjected to these restrictions and limitations. To this end, the following general boundary- and initial-conditions are specified for Eq. (1):

$$U_i(\mathbf{r})T(\mathbf{r}, t) \equiv [A_i(\mathbf{r})\mathbf{n}_i \cdot \nabla + B_i(\mathbf{r})]T(\mathbf{r}, t) = f_i(\mathbf{r}, t), \qquad \mathbf{r} \text{ on } S_i, \quad t > 0 \tag{2}$$

$$T(\mathbf{r}, t) = F(\mathbf{r}), \qquad \mathbf{r} \text{ in } R \text{ and on } S_i, \quad t = 0 \tag{3}$$

where $i = 1, 2, \cdots, q$; q is the number of co-ordinate surfaces, S_i, bounding R; $A_i(\mathbf{r}) \geq 0$, $B_i(\mathbf{r}) \geq 0$, $f_i(\mathbf{r}, t)$ are prescribed functions defined on S_i; $F(\mathbf{r})$ is the prescribed initial temperature field; $\mathbf{n}_i$ is the outward unit normal on S_i; and ∇ is the gradient vector in $\mathbf{r}$-space.

With $B_i(\mathbf{r}) \not\equiv 0$ for all $i = 1, 2, \cdots, q$, simultaneously, the solution to the system of Eqs. (1), (2) and (3) is [2]

$$\begin{aligned} T(\mathbf{r}, t) = \sum_{j=0}^{q} T_{0j}(\mathbf{r}, t) + \sum_{m=1}^{\infty} C_m\phi_m(\mathbf{r}) \exp(-\lambda_m^2\kappa t)\biggl\{\int_R \phi_m(\mathbf{r})F(\mathbf{r})\, dV \\ - \sum_{j=0}^{q}\left[\int_R \phi_m(\mathbf{r})T_{0j}(\mathbf{r}, 0)\, dV + \int_0^t \exp(\lambda_m^2\kappa\tau)\int_R \phi_m(\mathbf{r})\frac{\partial T_{0j}(\mathbf{r}, \tau)}{\partial\tau}\, dV\, d\tau\right]\biggr\} \end{aligned} \tag{4a}$$

where

$$(\nabla^2 + \lambda_m^2)\phi_m(\mathbf{r}) = 0, \qquad \mathbf{r} \text{ in } R, \tag{5a}$$

$$U_i(\mathbf{r})\phi_m(\mathbf{r}) = 0, \qquad \mathbf{r} \text{ on } S_i, \tag{5b}$$

$$\frac{1}{C_m} = \int_R \phi_m^2(\mathbf{r})\, dV \tag{5c}$$

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and the pseudo-steady temperature distributions of order zero, $T_{0j}(\mathbf{r}, t)$, are the solutions to the system

$$\left.\begin{aligned} \nabla^2 T_{0j}(\mathbf{r}, t) + \frac{\delta_{0j}}{K} Q(\mathbf{r}, t) &= 0, \qquad \mathbf{r} \text{ in } R \\ U_i(\mathbf{r}) T_{0j}(\mathbf{r}, t) &= \delta_{ij} f_i(\mathbf{r}, t), \qquad \mathbf{r} \text{ on } S_i \end{aligned}\right\} (i = 1, 2, \cdots, q; \quad j = 0, 1, \cdots, q) \tag{6a, 6b}$$

and have the eigenfunction expansion

$$T_{0j}(\mathbf{r}, t) = \frac{1}{K} \sum_{m=1}^{\infty} \frac{C_m}{\lambda_m^2} \phi_m(\mathbf{r}) \left[\delta_{0j} \int_R \phi_m(\mathbf{r}) Q(\mathbf{r}, t)\, dV + \delta_{ij} K \int_{S_j} \frac{\phi_m(\mathbf{r})}{A_j(\mathbf{r})} f_j(\mathbf{r}, t)\, dS_j \right]$$
$$(i = 1, 2, \cdots, q; \quad j = 0, 1, \cdots, q) \tag{6c}$$

δ_{0j} and δ_{ij} denoting Kronecker delta.

The solution (4a) can be expressed in the equivalent form of

$$\begin{aligned} T(\mathbf{r}, t) = \sum_{j=0}^{q} T_{0j}(\mathbf{r}, t) + \sum_{m=1}^{\infty} C_m \phi_m(\mathbf{r}) \exp(-\lambda_m^2 \kappa t) \Bigg\{ \int_R \phi_m(\mathbf{r}) F(\mathbf{r})\, dV \\ - \frac{1}{\lambda_m^2} \left[\frac{1}{K} \int_R \phi_m(\mathbf{r}) Q(\mathbf{r}, 0)\, dV + \sum_{i=1}^{q} \int_{S_i} \frac{\phi_m(\mathbf{r})}{A_i(\mathbf{r})} f_i(\mathbf{r}, 0)\, dS_i \right] \\ - \frac{1}{\lambda_m^2} \int_0^t \exp(\lambda_m^2 \kappa \tau) \left[\frac{1}{K} \int_R \phi_m(\mathbf{r}) \frac{\partial Q(\mathbf{r}, \tau)}{\partial \tau}\, dV + \sum_{i=1}^{q} \int_{S_i} \frac{\phi_m(\mathbf{r})}{A_i(\mathbf{r})} \frac{\partial f_i(\mathbf{r}, \tau)}{\partial \tau}\, dS_i \right] d\tau \Bigg\}. \end{aligned} \tag{4b}$$

An alternate expression for $T(\mathbf{r}, t)$ follows from (4b) as

$$\begin{aligned} T(\mathbf{r}, t) = \sum_{m=1}^{\infty} C_m \phi_m(\mathbf{r}) \exp(-\lambda_m^2 \kappa t) \Bigg\{ \int_R \phi_m(\mathbf{r}) F(\mathbf{r})\, dV \\ + \frac{\kappa}{K} \int_0^t \exp(\lambda_m^2 \kappa \tau) \left[\int_R \phi_m(\mathbf{r}) Q(\mathbf{r}, \tau)\, dV + K \sum_{i=1}^{q} \int_{S_i} \frac{\phi_m(\mathbf{r})}{A_i(\mathbf{r})} f_i(\mathbf{r}, \tau)\, dS_i \right] d\tau \Bigg\}. \end{aligned} \tag{4c}$$

The expression (4c), although simpler in form than the expression (4b), is not uniformly convergent except when $f_i(\mathbf{r}, t) = 0$ for all i. On the other hand, under the conditions cited in [2], the expressions (4a) and (4b) converge uniformly and are well suited for engineering purposes in view of their more rapid convergence. Implicit in (4a) and (4b) is the assumption of the existence of first order derivatives of the source functions, $f_i(\mathbf{r}, t)$, $Q(\mathbf{r}, t)$, with respect to t. Letting $f_i(\mathbf{r}, t) = F(\mathbf{r}) = 0$ in the expression (4c), and taking $Q(\mathbf{r}, t)$ as being separable in $\mathbf{r}$ and t, equation (4c) reduces to the result (2) given in [1].

The particular case where

$$U_i(\mathbf{r}) T(\mathbf{r}, t) \equiv K(\mathbf{n}_i \cdot \nabla) T(\mathbf{r}, t) = f_i(\mathbf{r}, t), \qquad \mathbf{r} \text{ on } S_i, \quad t > 0 \tag{7}$$

for all $i = 1, 2, \cdots, q$, simultaneously, deserves special attention. The general solution to the system of (1), (7) and (3) can be shown to be [3]

$$\begin{aligned} T(\mathbf{r}, t) = \frac{1}{V} \int_R F(\mathbf{r})\, dV + \sum_{j=0}^{q} [\Omega_j(t) + T_{0j}(\mathbf{r}, t)] \\ + \sum_{m=1}^{\infty} C_m \phi_m(\mathbf{r}) \exp(-\lambda_m^2 \kappa t) \Bigg\{ \int_R \phi_m(\mathbf{r}) \left[F(\mathbf{r}) - \sum_{j=0}^{q} T_{0j}(\mathbf{r}, 0) \right] dV \\ - \sum_{j=0}^{q} \int_0^t \exp(\lambda_m^2 \kappa \tau) \int_R \phi_m(\mathbf{r}) \frac{\partial T_{0j}(\mathbf{r}, \tau)}{\partial \tau}\, dV\, d\tau \Bigg\} \end{aligned} \tag{8a}$$

where V is the volume of the finite region R; $\phi_m(\mathbf{r})$ and λ_m are, respectively, the eigenfunctions and the positive eigenvalues of the system of (5a) and (5b), where $U_i(\mathbf{r})$ is now defined in (7); and

$$\Omega_j(t) = \frac{\kappa}{KV}\left[\delta_{0j}\int_0^t\int_R Q(\mathbf{r},\tau)\,dV\,d\tau + \delta_{ij}\int_0^t\int_{S_i} f_i(\mathbf{r},\tau)\,dS_i\,d\tau\right]$$

$$(i = 1, 2, \cdots, q;\quad j = 0, 1, \cdots, q). \tag{9}$$

The pseudo-steady temperature distributions of order zero, $T_{0j}(\mathbf{r}, t)$, are now to be determined from the system of

$$\nabla^2 T_{0j}(\mathbf{r}, t) + \frac{\delta_{0j}}{K} Q(\mathbf{r}, t) = \frac{1}{\kappa}\frac{d\Omega_j(t)}{dt}, \quad \mathbf{r} \text{ in } R \tag{10a}$$

$$K(\mathbf{n}_i\cdot\nabla)T_{0j}(\mathbf{r}, t) = \delta_{ij} f_i(\mathbf{r}, t), \quad \mathbf{r} \text{ on } S_i \quad \begin{pmatrix} i = 1, 2, \cdots, q; \\ j = 0, 1, \cdots, q \end{pmatrix} \tag{10b}$$

$$\int_R T_{0j}(\mathbf{r}, t)\,dV = 0 \tag{10c}$$

where t plays the role of a parameter, and the eigenfunction representation for $T_{0j}(\mathbf{r}, t)$ is

$$T_{0j}(\mathbf{r}, t) = \frac{1}{K}\sum_{m=1}^{\infty}\frac{C_m}{\lambda_m^2}\phi_m(\mathbf{r})\left[\delta_{0j}\int_R \phi_m(\mathbf{r})Q(\mathbf{r}, t)\,dV + \delta_{ij}\int_{S_i}\phi_m(\mathbf{r})f_i(\mathbf{r}, t)\,dS_i\right]$$

$$(i = 1, 2, \cdots, q; j = 0, 1, \cdots, q). \tag{11}$$

Whenever the $T_{0j}(\mathbf{r}, t)$ functions can be determined directly from the system (10), the expression (11) constitutes a set of summation formulae.

By use of (9) and (11) the solution (8a) can be expressed in the equivalent form of

$$\begin{aligned} T(\mathbf{r}, t) = {} & \frac{1}{V}\int_R F(\mathbf{r})\,dV + \frac{\kappa}{KV}\int_0^t\left[\int_R Q(\mathbf{r},\tau)\,dV + \sum_{i=1}^{q}\int_{S_i} f_i(\mathbf{r},\tau)\,dS_i\right]d\tau \\ & + \sum_{j=0}^{q} T_{0j}(\mathbf{r}, t) + \sum_{m=1}^{\infty} C_m\phi_m(\mathbf{r})\exp(-\lambda_m^2\kappa t)\left\{\int_R \phi_m(\mathbf{r})F(\mathbf{r})\,dV\right. \\ & - \frac{1}{K\lambda_m^2}\left[\int_R \phi_m(\mathbf{r})Q(\mathbf{r}, 0)\,dV + \sum_{i=1}^{q}\int_{S_i}\phi_m(\mathbf{r})f_i(\mathbf{r}, 0)\,dS_i\right] \\ & \left. - \int_0^t \frac{\exp(\lambda_m^2\kappa\tau)}{K\lambda_m^2}\left[\int_R \phi_m(\mathbf{r})\frac{\partial Q(\mathbf{r},\tau)}{\partial\tau}\,dV + \sum_{i=1}^{q}\int_{S_i}\phi_m(\mathbf{r})\frac{\partial f_i(\mathbf{r},\tau)}{\partial\tau}\,dS_i\right]d\tau\right\} \end{aligned} \tag{8b}$$

where the source functions $f_i(\mathbf{r}, t)$ and $Q(\mathbf{r}, t)$ appear explicitly.

A further alternate expression for $T(\mathbf{r}, t)$ follows from (8b) and (11) as

$$\begin{aligned} T(\mathbf{r}, t) = {} & \frac{1}{V}\int_R F(\mathbf{r})\,dV + \frac{\kappa}{KV}\int_0^t\left[\int_R Q(\mathbf{r},\tau)\,dV + \sum_{i=1}^{q}\int_{S_i} f_i(\mathbf{r},\tau)\,dS_i\right]d\tau \\ & + \sum_{m=1}^{\infty} C_m\phi_m(\mathbf{r})\exp(-\lambda_m^2\kappa t)\left\{\int_R \phi_m(\mathbf{r})F(\mathbf{r})\,dV\right. \\ & \left. + \frac{\kappa}{K}\int_0^t \exp(\lambda_m^2\kappa\tau)\left[\int_R \phi_m(\mathbf{r})Q(\mathbf{r},\tau)\,dV + \sum_{i=1}^{q}\int_{S_i}\phi_m(\mathbf{r})f_i(\mathbf{r},\tau)\,dS_i\right]d\tau\right\} \end{aligned} \tag{8c}$$

in which the pseudo-steady solutions $T_{0i}(\mathbf{r}, t)$ do not appear. Again, the convergence of (8a) and (8b) is much faster than that of (8c) which does not converge uniformly unless $f_i(\mathbf{r}, t) = 0$.

References

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