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SLIGHTLY DAMPED LIBRATIONS*

By J. A. MORRISON (*Bell Telephone Laboratories, Inc., Murray Hill, New Jersey*)

Abstract. The application of a generalized method of averaging to the problem of slightly damped librations in a perturbed one degree of freedom system is considered. The librations are those arising in the critical case in which the rate of change of the phase in the unperturbed system has a zero. The perturbed system is reduced to a form suitable for the application of the generalized method of averaging, and the first order averaged equation is derived for the slow rate of change of the amplitude of the librations, due to the damping.

1. Introduction. Consider the system of two first order differential equations

$$\frac{dx}{dt} = \epsilon f(x, \varphi; \epsilon); \qquad \frac{d\varphi}{dt} = \omega(x) + \epsilon g(x, \varphi; \epsilon), \tag{1.1}$$

where $\epsilon > 0$ is a small parameter and

$$\begin{aligned} f(x, \varphi; \epsilon) &= f^{(1)}(x, \varphi) + \epsilon f^{(2)}(x, \varphi) + \cdots ; \\ g(x, \varphi; \epsilon) &= g^{(1)}(x, \varphi) + \epsilon g^{(2)}(x, \varphi) + \cdots . \end{aligned} \tag{1.2}$$

If $\omega(x) \neq 0$ and f and g are periodic in φ, then the method of averaging in the case of a rapidly rotating phase [1] may be applied to obtain an asymptotic solution to the system (1.1). We wish to consider the critical case in which $\omega(c) = 0$, but $\omega'(c) \neq 0$. It is further supposed that $f^{(1)}(c, \gamma) = 0$ and

$$0 < \lambda \leq -\omega'(c)f^{(1)}(c, \varphi)/(\varphi - \gamma) \leq \Lambda, \tag{1.3}$$

for $-\psi_* \leq (\varphi - \gamma) \leq \psi^*$, say, so that we have the case of a libration, as will be evident later. Note that we do not have to assume that f and g are periodic in φ. The case in which (1.1) is a Hamiltonian system, and hence undamped, has been dealt with by Gormally [2]. Here we consider librations which, in general, are damped.

For $\epsilon > 0$ there is a stationary point

$$\begin{aligned} x &= \xi(c, \gamma; \epsilon) = c + \epsilon\xi^{(1)}(c, \gamma) + \epsilon^2\xi^{(2)}(c, \gamma) + \cdots , \\ \varphi &= \rho(c, \gamma; \epsilon) = \gamma + \epsilon\rho^{(1)}(c, \gamma) + \epsilon^2\rho^{(2)}(c, \gamma) + \cdots , \end{aligned} \tag{1.4}$$

of the system (1.1), satisfying

$$f(\xi, \rho; \epsilon) = 0; \qquad \omega(\xi) + \epsilon g(\xi, \rho; \epsilon) = 0. \tag{1.5}$$

Substituting (1.4) into (1.5), and expanding in powers of ϵ using (1.2), we find, in par-

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ticular, that

$$\omega'(c)\xi^{(1)} + g^{(1)}(c, \gamma) = 0, \tag{1.6}$$

$$\xi^{(1)} \frac{\partial f^{(1)}}{\partial x}(c, \gamma) + \rho^{(1)} \frac{\partial f^{(1)}}{\partial \varphi}(c, \gamma) + f^{(2)}(c, \gamma) = 0. \tag{1.7}$$

These equations determine $\xi^{(1)}$ and $\rho^{(1)}$ since $\omega'(c) \neq 0$ and $(\partial f^{(1)}/\partial\varphi)(c, \gamma) \neq 0$, and the higher order terms in (1.4) may be determined by iteration.

Let $\mu = \epsilon^{1/2}$ and make the transformations

$$x = [\xi(c, \gamma; \mu^2) + \mu y]; \qquad \varphi = [\rho(c, \gamma; \mu^2) + \psi]; \qquad \tau = \mu t. \tag{1.8}$$

Then the system (1.1) takes the form

$$\frac{dy}{d\tau} = h(\psi) + \mu F(y, \psi; \mu); \qquad \frac{d\psi}{d\tau} = \omega'(c)y + \mu G(y, \psi; \mu), \tag{1.9}$$

where

$$h(\psi) = f^{(1)}(c, \gamma + \psi), \tag{1.10}$$

$$F(y, \psi; \mu) = \{f[\xi(c, \gamma; \mu^2) + \mu y, \rho(c, \gamma; \mu^2) + \psi; \mu^2] - f^{(1)}(c, \gamma + \psi)\}/\mu, \tag{1.11}$$

and

$$G(y, \psi; \mu) = g[\xi(c, \gamma; \mu^2) + \mu y, \rho(c, \gamma; \mu^2) + \psi; \mu^2] + \{\omega[\xi(c, \gamma; \mu^2) + \mu y] - \mu\omega'(c)y\}/\mu^2. \tag{1.12}$$

Hence

$$\begin{aligned} F(y, \psi; \mu) &= F^{(1)}(y, \psi) + \mu F^{(2)}(y, \psi) + \cdots ; \\ G(y, \psi; \mu) &= G^{(1)}(y, \psi) + \mu G^{(2)}(y, \psi) + \cdots , \end{aligned} \tag{1.13}$$

where $F^{(l)}(y, \psi)$ and $G^{(l)}(y, \psi)$ are polynomials in y, and

$$F^{(l)}(0, 0) = 0; \qquad G^{(l)}(0, 0) = 0; \qquad h(0) = 0. \tag{1.14}$$

In particular, we have

$$F^{(1)}(y, \psi) = y \frac{\partial f^{(1)}}{\partial x}(c, \gamma + \psi), \tag{1.15}$$

and

$$G^{(1)}(y, \psi) = \left[\frac{y^2}{2}\omega''(c) + g^{(1)}(c, \gamma + \psi) - g^{(1)}(c, \gamma)\right], \tag{1.16}$$

using (1.6). We proceed to reduce the system (1.9) to a form suitable for the application of a generalized method of averaging.

2. Reduction in the odd case. In a previous paper [3] we discussed a generalized method of averaging, which could be applied directly to a perturbed vector system of differential equations of the form

$$\begin{aligned} \frac{dz_i}{dt} &= \mu Z_i^{(1)}(\mathbf{z}, \theta) + \mu^2 Z_i^{(2)}(\mathbf{z}, \theta) + \cdots ; \\ \frac{d\theta}{dt} &= \Omega^{(0)}(\mathbf{z}, \theta) + \mu\Omega^{(1)}(\mathbf{z}, \theta) + \mu^2\Omega^{(2)}(\mathbf{z}, \theta) + \cdots , \end{aligned} \tag{2.1}$$

where $Z_i^{(1)}(\mathbf{z}, \theta)$ and $\Omega^{(1)}(\mathbf{z}, \theta)$ are periodic in θ, with fixed period, and $\Omega^{(0)}(\mathbf{z}, \theta) \neq 0$ in the range of interest. Although this system may be reduced to a form appropriate for the application of the method of averaging in the case of a rapidly rotating phase, the transformation required is not, in general, explicit. On the other hand, our generalized method of averaging applied directly to the system (2.1) is an explicit procedure. We note here the first order averaged equations corresponding to the slowly varying quantities z_i, namely

$$\frac{dz_i}{dt} = \mu T_i^{(1)}(\mathbf{z}), \tag{2.2}$$

where

$$\langle 1/\Omega^{(0)}(\mathbf{z}, \theta)\rangle T_i^{(1)}(\mathbf{z}) = \langle Z_i^{(1)}(\mathbf{z}, \theta)/\Omega^{(0)}(\mathbf{z}, \theta)\rangle, \tag{2.3}$$

and the brackets denote averages over a period of θ, wherein $\mathbf{z}$ is held constant. We refer the reader to [3] for the development of the higher order approximations, and to [4] and [5] for two excellent review articles on averaging methods. We will now reduce the system (1.9) to a system of the form (2.1), with $\mathbf{z}$ a scalar.

We first consider the case $h(-\psi) = -h(\psi)$, where $h(\psi)$ is as given in (1.10), and let

$$\Phi(\beta) = -2\omega'(c) \int_0^\beta h(\psi)\, d\psi. \tag{2.4}$$

Note, from (1.3) and (1.10), that $\Phi(\beta)$ is a monotonic increasing function of β in the range $0 \leq \beta \leq \psi^*$ and, in the present case, $\Phi(-\beta) = \Phi(\beta)$. In a manner similar to that adopted in [3], where we discussed slightly damped nonlinear oscillations, we make the transformations

$$\psi = \alpha \sin\theta; \qquad \omega'(c)y = \cos\theta\{[\Phi(\alpha) - \Phi(\alpha\sin\theta)]\sec^2\theta\}^{1/2}, \tag{2.5}$$

where $\alpha > 0$ and the positive square root is to be taken. The quantity α may be regarded as the amplitude of the phase libration. It is a straightforward matter to verify that the system (1.9) transforms into

$$h(\alpha)\frac{d\alpha}{d\tau} = \mu[h(\psi)G - \cos\theta\{[\Phi(\alpha) - \Phi(\alpha\sin\theta)]\sec^2\theta\}^{1/2}F], \tag{2.6}$$

$$\alpha\frac{d\theta}{d\tau} = \{[\Phi(\alpha) - \Phi(\alpha\sin\theta)]\sec^2\theta\}^{1/2}\left[1 + \frac{\mu\sin\theta}{h(\alpha)}F\right] + \mu G\sec\theta[1 - \sin\theta\, h(\alpha\sin\theta)/h(\alpha)]. \tag{2.7}$$

Note that the factor of G in (2.7) remains finite as $\cos\theta \to 0$. In the case of damped librations $\alpha \to 0$, and it is noted, from (1.14) and (2.4), that $d\theta/d\tau$ remains finite for $\alpha \to 0$. From (1.13) and (2.5) it is seen that (2.6) and (2.7) are in a form suitable for the application of the generalized method of averaging.

From (1.13), (2.1)–(2.3) and (2.5)–(2.7), the first order averaged equation for α is

$$h(\alpha)\frac{d\alpha}{d\tau}\int_0^{2\pi}\{[\Phi(\alpha) - \Phi(\alpha\sin\theta)]\sec^2\theta\}^{-1/2}\,d\theta = \mu P(\alpha)$$
$$\equiv \mu\int_0^{2\pi}[h(\alpha\sin\theta)G^{(1)}\{[\Phi(\alpha) - \Phi(\alpha\sin\theta)]\sec^2\theta\}^{-1/2} - F^{(1)}\cos\theta]\,d\theta. \tag{2.8}$$

Substituting from (1.15), (1.16) and (2.5), it is found that

$$P(\alpha) = \int_0^{2\pi} h(\alpha \sin \theta) g^{(1)}(c, \gamma + \alpha \sin \theta) \{[\Phi(\alpha) - \Phi(\alpha \sin \theta)] \sec^2 \theta\}^{-1/2} \, d\theta$$

$$- \frac{1}{\omega'(c)} \int_0^{2\pi} \cos^2 \theta \frac{\partial f^{(1)}}{\partial x} (c, \gamma + \alpha \sin \theta) \{[\Phi(\alpha) - \Phi(\alpha \sin \theta)] \sec^2 \theta\}^{1/2} \, d\theta, \tag{2.9}$$

the terms involving $\omega''(c)$ and $g^{(1)}(c, \gamma)$ vanishing identically, since $h(\alpha \sin \theta)$ and $\Phi(\alpha \sin \theta)$ are, respectively, odd and even in θ. But, from (2.4),

$$\frac{d}{d\theta} [\cos \theta \{[\Phi(\alpha) - \Phi(\alpha \sin \theta)] \sec^2 \theta\}^{1/2}]$$

$$= \alpha \omega'(c) h(\alpha \sin \theta) \{[\Phi(\alpha) - \Phi(\alpha \sin \theta)] \sec^2 \theta\}^{-1/2}. \tag{2.10}$$

Hence, integrating by parts,

$$\int_0^{2\pi} h(\alpha \sin \theta) g^{(1)}(c, \gamma + \alpha \sin \theta) \{[\Phi(\alpha) - \Phi(\alpha \sin \theta)] \sec^2 \theta\}^{-1/2} d\theta$$

$$= -\frac{1}{\omega'(c)} \int_0^{2\pi} \cos^2 \theta \frac{\partial g^{(1)}}{\partial \varphi} (c, \gamma + \alpha \sin \theta) \{[\Phi(\alpha) - \Phi(\alpha \sin \theta)] \sec^2 \theta\}^{1/2} \, d\theta. \tag{2.11}$$

From (2.8), (2.9) and (2.11), noting that the integrands are even about $\theta = \pi/2$, we obtain finally

$$h(\alpha) \frac{d\alpha}{d\tau} \int_{-\pi/2}^{\pi/2} \{[\Phi(\alpha) - \Phi(\alpha \sin \theta)] \sec^2 \theta\}^{-1/2} \, d\theta = -\frac{\mu}{\omega'(c)} \int_{-\pi/2}^{\pi/2} \left[\frac{\partial f^{(1)}}{\partial x} (c, \gamma + \alpha \sin \theta) \right.$$

$$\left. + \frac{\partial g^{(1)}}{\partial \varphi} (c, \gamma + \alpha \sin \theta) \right] \{[\Phi(\alpha) - \Phi(\alpha \sin \theta)] \sec^2 \theta\}^{1/2} \cos^2 \theta \, d\theta. \tag{2.12}$$

We comment on the special case in which the system (1.1) is a Hamiltonian one. Then, in particular, using (1.2),

$$f^{(1)}(x, \varphi) = -\frac{\partial H^{(1)}}{\partial \varphi}; \qquad g^{(1)}(x, \varphi) = \frac{\partial H^{(1)}}{\partial x}. \tag{2.13}$$

Hence, from (2.12), the first order averaged equation for the amplitude α gives $d\alpha/d\tau = 0$, as is to be expected. In the next section we obtain the result analogous to (2.12), in the case that $h(\psi)$ is not an odd function of ψ.

3. Reduction in the general case. In the general case, when $h(\psi)$ is not an odd function of ψ, the reduction of the system (1.9) to a form suitable for the application of the generalized method of averaging is not completely explicit. Nevertheless, it is feasible to apply the averaging method. We may still introduce the quantity $\Phi(\beta)$ as in (2.4), but it will no longer be an even function of β. We now make the transformations

$$\omega'(c) y = k \cos \chi; \qquad \psi[\Phi(\psi)/\psi^2]^{1/2} = k \sin \chi; \qquad \omega'(c)\tau = s, \tag{3.1}$$

where the positive square root is to be taken. Then, as is readily verified, the system (1.9) transforms into

$$\frac{dk}{ds} = \mu \left[F \cos \chi - \frac{h(\psi)}{k} G \right], \tag{3.2}$$

$$k\frac{d\chi}{ds} = -\left\{\frac{h(\psi)}{\sin\chi} + \mu\left[F\sin\chi + \frac{h(\psi)G}{k\sin\chi}\cos\chi\right]\right\}. \tag{3.3}$$

From (1.13) and (3.1) it is seen that (3.2) and (3.3) are in a form suitable for the application of the generalized method of averaging since, implicitly at least, $\psi = v(k\sin\chi)$.

The first order averaged equation for k is, from (1.13), (2.1)–(2.3), (3.2) and (3.3),

$$\frac{dk}{ds}\int_0^{2\pi}\frac{\sin\chi}{h(\psi)}\,d\chi = \mu\int_0^{2\pi}\left[F^{(1)}\cos\chi - \frac{h(\psi)}{k}G^{(1)}\right]\frac{\sin\chi}{h(\psi)}\,d\chi. \tag{3.4}$$

In view of the implicit relationship expressing ψ as a function of χ, it is convenient to change the variable of integration in (3.4) from χ to ψ. But, from (2.4) and (3.1),

$$\frac{\sin\chi\,d\chi}{h(\psi)} = -\frac{\omega'(c)\,d\psi}{k^2\cos\chi} = -\frac{\omega'(c)\,d\psi}{k[k^2 - \Phi(\psi)]^{1/2}}. \tag{3.5}$$

Now ψ oscillates between the values $-\psi_*(k)$ and $\psi^*(k)$, where

$$\Phi[\psi^*(k)] = k^2 = \Phi[-\psi_*(k)], \tag{3.6}$$

and $[k^2 - \Phi(\psi)]^{1/2}$ is positive during one half of the cycle and negative during the other half. We may rewrite (3.4), using (1.15), (1.16) and (3.1), in the form

$$k\frac{dk}{ds}\oint[k^2 - \Phi(\psi)]^{-1/2}\,d\psi = \frac{\mu}{\omega'(c)}\oint\frac{\partial f^{(1)}}{\partial x}(c, \gamma + \psi)[k^2 - \Phi(\psi)]^{1/2}\,d\psi$$
$$- \mu\oint h(\psi)\left\{\frac{\omega''(c)}{2[\omega'(c)]^2}[k^2 - \Phi(\psi)]^{1/2} + [g^{(1)}(c, \gamma + \psi) - g^{(1)}(c, \gamma)][k^2 - \Phi(\psi)]^{-1/2}\right\}d\psi, \tag{3.7}$$

and replace each integral over the cycle by twice the integral from $-\psi_*(k)$ to $\psi^*(k)$.

But, from (2.4), $2\omega'(c)h(\psi) = -\Phi'(\psi)$. Hence

$$\omega'(c)\int_{-\psi_*(k)}^{\psi^*(k)} h(\psi)[k^2 - \Phi(\psi)]^{-1/2}\,d\psi = [[k^2 - \Phi(\psi)]^{1/2}]_{-\psi_*(k)}^{\psi^*(k)} = 0, \tag{3.8}$$

from (3.6), and similarly

$$\int_{-\psi_*(k)}^{\psi^*(k)} h(\psi)[k^2 - \Phi(\psi)]^{1/2}\,d\psi = 0. \tag{3.9}$$

Also, integrating by parts,

$$\int_{-\psi_*(k)}^{\psi^*(k)} h(\psi)g^{(1)}(c, \gamma + \psi)[k^2 - \Phi(\psi)]^{-1/2}\,d\psi$$
$$= -\frac{1}{\omega'(c)}\int_{-\psi_*(k)}^{\psi^*(k)}\frac{\partial g^{(1)}}{\partial\varphi}(c, \gamma + \psi)[k^2 - \Phi(\psi)]^{1/2}\,d\psi. \tag{3.10}$$

Thus, (3.7) becomes

$$k\frac{dk}{ds}\int_{-\psi_*(k)}^{\psi^*(k)}[k^2 - \Phi(\psi)]^{-1/2}\,d\psi$$
$$= \frac{\mu}{\omega'(c)}\int_{-\psi_*(k)}^{\psi^*(k)}\left[\frac{\partial f^{(1)}}{\partial x}(c, \gamma + \psi) + \frac{\partial g^{(1)}}{\partial\varphi}(c, \gamma + \psi)\right][k^2 - \Phi(\psi)]^{1/2}\,d\psi. \tag{3.11}$$

Now let us return to the case when $h(\psi)$ is an odd function of ψ, so that $\Phi(\psi)$ is an even function of ψ. Then, from (2.5), (3.1) and (3.6),

$$k^2 = \Phi(\alpha); \qquad \psi^*(k) = \alpha = \psi_*(k). \tag{3.12}$$

Also, using (2.4),

$$k \frac{dk}{ds} = -h(\alpha) \frac{d\alpha}{d\tau}. \tag{3.13}$$

The substitution $\psi = \alpha \sin \theta$ in (3.11) then leads to (2.12). It was considered worthwhile, however, to treat the case when $h(\psi)$ is an odd function of ψ separately, since the reduction of the system (1.9), to a form suitable for the application of the generalized method of averaging, may then be carried out explicitly.

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