ON THE CONSTRUCTION OF HERMITIAN FROM LAGRANGIAN DIFFERENCE APPROXIMATIONS*

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Abstract. It is shown how simple finite difference approximations to the Laplace operator in two- or three-dimensions can be combined to construct Hermitian finite difference approximations to the two- or three-dimensional Poisson equation $\Delta u = f$.

1. Introduction. This note is concerned with finite difference approximations to Poisson's equation, $\Delta u = f$, where Δ denotes the Laplace operator and f is a given function. These finite difference approximations are called Lagrangian or Hermitian according to whether they involve only one or several values of f. (Collatz [1], who first stressed the accuracy obtainable with Hermitian approximations, coined the term "Mehrstellenverfahren".)

Whereas Lagrangian approximations of various degrees of precision are readily established, the construction of Hermitian approximations is less direct. Indeed, the Taylor expansions of u and $\Delta u = f$ at a given node of a square or cubic grid must be evaluated for the neighboring nodes, and the resulting expressions must be combined in such a manner that all terms below a certain degree in the mesh width h cancel out. A method of achieving this "Taylor cancellation" (called "Taylor Abgleich" by Collatz [1]) has been described by Meister and Prager [2]. While this method is straightforward, it requires a certain amount of algebraic work, which increases rapidly with the desired degree of precision.

The present note is meant to draw attention to the way in which readily obtainable Lagrangian approximations to Δu can be combined to yield approximations to $\Delta^2 u$ and Hermitian approximations to the equation $\Delta u = f$.

2. Lagrangian approximations to Δu and $\Delta^2 u$. Figure 1 shows the labelling of the nodes of a two-dimensional square grid in the neighborhood of the typical node 0. Note that there is only one node labelled 0, but there are four nodes each labelled 1, 2, 3. The following discussion is restricted to symmetric difference operators in which the values of the function u at equally labelled nodes appear with the same weight. The sum of the values of u at all nodes labelled i will be denoted by u_i. Similarly, Δu_i will denote the sum of the values of Δu at all nodes labelled i, and so on.

With this notation, the most familiar finite difference approximations to Δu_0 may be written in the forms

$$h^2 \Delta u_0 = -4u_0 + u_1 + h^4 Q u_0 + O(h^6), \tag{1}$$

and

$$12h^2 \Delta u_0 = -60u_0 + 16u_1 - u_2 + h^6 R u_0 + O(h^8), \tag{2}$$

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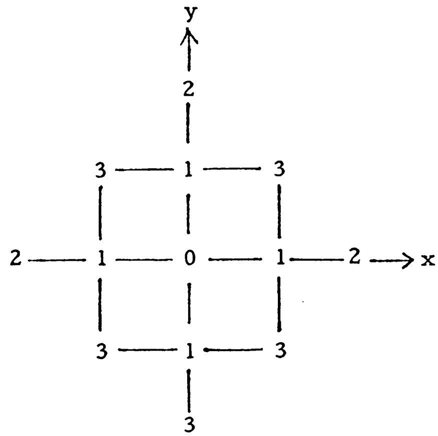

FIG. 1. Nodes of square grid and nodes in plane $z = 0$ of cubic grid.

where

$$Q = -\frac{1}{12}\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4}\right), \qquad R = \frac{2}{15}\left(\frac{\partial^6}{\partial x^6} + \frac{\partial^6}{\partial y^6}\right), \tag{3}$$

the coordinate axis being parallel to the grid lines.

Applying (1) to determine the sum Δu_1 of the values of Δu at the four nodes labelled 1, we find

$$h^2 \Delta u_1 = 4u_0 - 4u_1 + u_2 + 2u_3 + h^4 Q u_1 + O(h^6). \tag{4}$$

A finite difference expression for $h^4 \Delta^2 u_0$ may now be obtained by evaluating the Laplacian of $h^4 \Delta u$ at the node 0 in accordance with (1) and using (4). Thus,

$$\begin{aligned} h^4 \Delta^2 u_0 &= -4h^2 \Delta u_0 + h^2 \Delta u_1 + h^6 \Delta Q u_0 + O(h^8) \\ &= 20u_0 - 8u_1 + u_2 + 2u_3 + h^4 Q[-4u_0 + u_1] + h^6 \Delta Q u_0 + O(h^8). \end{aligned} \tag{5}$$

By (1), the bracket in (5) equals $h^2 \Delta u_0$ to within terms of the order h^4. Since $\Delta Q = Q\Delta$, we find

$$h^4 \Delta^2 u_0 = 20u_0 - 8u_1 + u_2 + 2u_3 + 2h^6 Q \Delta u_0 + O(h^8). \tag{6}$$

Analogous formulas in three dimensions may be obtained by the same technique. When the nodes are labelled as shown in Figs. 1 and 2, the simplest finite difference approximations to Δu_0 are

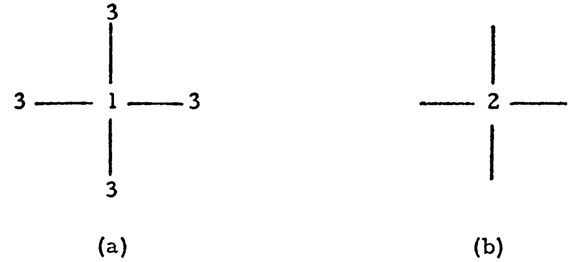

FIG. 2. Nodes of cubic grid: (a) Nodes in planes $z = \pm h$. (b) Nodes in plane $z = \pm 2h$. For nodes in plane $z = 0$, see Fig. 1.

$$h^2\Delta u_0 = -6u_0 + u_1 + h^4 S u_0 + O(h^6), \tag{7}$$

and

$$12h^2\Delta u_0 = -90u_0 + 16u_1 - u_2 + h^6 T u_0 + O(h^8), \tag{8}$$

where

$$S = -\frac{1}{12}\left(\frac{\partial^4}{\partial x^4} + \frac{\partial^4}{\partial y^4} + \frac{\partial^4}{\partial z^4}\right), \qquad T = \frac{2}{15}\left(\frac{\partial^6}{\partial x^6} + \frac{\partial^6}{\partial y^6} + \frac{\partial^6}{\partial z^6}\right) \tag{9}$$

Using (7) in the manner in which (1) was used to derive (6), one readily obtains

$$h^4\Delta^2 u_0 = 42u_0 - 12u_1 + u_2 + 2u_3 + 2h^6 S\Delta u_0 + O(h^8). \tag{10}$$

3. Hermitian approximations to $\Delta u = f$. Whereas formula (2) for Δu_0 has a higher degree of precision than formula (1), it involves the values of u at the points 2 that have the distance $2h$ from the point 0. This means that even for a rectangular domain formula (2) cannot be used at interior points with the distance h from the boundary. To avoid the necessity of constructing special formulas for these points near the boundary without sacrifice in the degree of precision, we may eliminate u_2 by combining the finite difference approximation to $12\Delta u = 12f$ obtained from (2) with the one to $\Delta^2 u = \Delta f$ obtained from (6). Replacing Δu_0 in (2) and (6) by f_0 and adding the resulting equations, we find

$$h^4\Delta f_0 + 12h^2 f_0 = -40u_0 + 8u_1 + 2u_3 + h^6(2Q\Delta + R)u_0 + O(h^8). \tag{11}$$

We now use the approximation (1) for Δf_0 and note that $Qf_0 = Q\Delta u_0$. Thus,

$$8h^2 f_0 + h^2 f_1 = -40u_0 + 8u_1 + 2u_3 + h^6(Q\Delta + R)u_0 + O(h^8). \tag{12}$$

The same technique may be applied in three dimensions. Eliminating u_2 by adding the finite difference approximations to $12\Delta u = 12f$ and $\Delta^2 u = \Delta f$ obtained from (8) and (10), we find

$$h^4\Delta f_0 + 12h^2 f_0 = -48u_0 + 4u_1 + 2u_3 + h^6(2S\Delta + T)u_0 + O(h^8). \tag{13}$$

Finally, evaluating Δf_0 by (7) and observing that $Sf_0 = S\Delta u_0$, we obtain

$$6h^2 f_0 + h^2 f_1 = -48u_0 + 4u_1 + 2u_3 + h^6(S\Delta + T)u_0 + O(h^8). \tag{14}$$

Formulas (12) and (14) are well known; the first is due to Collatz [1] and the second to Albrecht [3]. The customary derivation of these formulas, however, is less direct than the one presented above.

Hermitian approximations of higher degrees of precision can be constructed in a similar manner.

References

1. L. Collatz, *Numerische Behandlung von Differentialgleichungen*, Springer, Berlin, 1951; see also *Das Mehrstellenverfahren bei Plattenaufgaben*, Z. Angew. Math. Mech. **30**, 385-388, and *Einige neuere Forschungen über numerische Behandlung von Differentialgleichungen*, Z. Angew. Math. Mech. **31**, 230-236 (1951)
2. B. Meister and W. Prager, *On the construction of symmetric difference operators for square and cubic lattices*, Z. Angew. Math. Phys. **16**, 403-410 (1965)
3. J. Albrecht, *Taylor-Entwicklungen und finite Ausdrücke für Δu und $\Delta\Delta u$*. Z. Angew. Math. Mech. **33**, 41-48 (1953)