MEANS, VARIANCES, AND COVARIANCES FOR LASER BEAM PROPAGATION THROUGH A RANDOM MEDIUM*

BY

ROBERT A. SCHMELTZER

Electro-Optical Laboratory, Autonetics Division,
North American Aviation, Inc., Anaheim, California

Abstract. Wave propagation in a random continuous medium is studied by solving the stochastic wave equation with a random function for the refractive index coefficient. By the application of the Rytov transformation, an equivalent spatial form of the nonlinear Riccati equation is obtained which is then solved by means of an iteration scheme. The statistical properties of the propagated wave are then computed for the case of a coherent focused source with a Gaussian amplitude distribution. These formulas contain, as limiting sub-cases, the results of previous analyses for the spherical and plane wave. More generally, they describe the propagation of a laser beam.

1. Introduction. A fundamental limitation to communication by coherent light transmission through the atmosphere is the unavoidable effects created by normal atmospheric turbulence. The presence of random inhomogenietics in the atmosphere causes a random fluctuation of the refractive index which in turn, distorts, bends, and impresses undesirable modulation on laser light, until eventually, all coherence is destroyed. The present paper is devoted primarily to the mathematics for determining the statistical behavior of an initially coherent beam of focused light propagated in a medium which, while isotropic, is characterized by having a random point function for its refractive index. Wave propagation in such a medium is usually referred to as turbulent scattering. We shall not be concerned with the theoretics of the statistical mechanics of the random medium in view of the confirmed theory of Kolomogoroff [1]. Our concern is with the stochastic aspects of the problem. Wave propagation in a random continuous medium is governed by a stochastic wave equation, whose index of refraction coefficient is a continuous random point function, and characterizes the transmission medium. The solution to the equation is obtained by means of an iteration scheme and is expressed in the form of a power series in a perturbation parameter.

The theory is applied to the study of the propagation of a focused laser beam through a turbulent atmosphere. The statistical properties of the solution are computed, viz., means, variances, covariances, and structure functions. The structure function has been amply demonstrated to be a highly significant quantity for describing the random perturbations of an optical wavefront, and for predicting the performance of optical systems [2]. Such parameters as angular resolution, angular and amplitude scintillations, and efficiency of coherent heterodyne detectors all depend directly upon the structure function of the distorted optical wave front [3].

The results are kept fairly general by describing the beam by parameters such as initial spot diameter, beam spread (a function of path length), and an arbitrary radius of curvature of the focused wavefront such as would be produced by the action of an

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ideal lens. The formulas, therefore, contain as limiting special cases, the results of previous analyses for the plane wave and spherical wave [1], [4], [5]. The results are found to depend on a knowledge of the covariance function for the index of refraction along the path of propagation of the principal ray.

It is believed that the results achieved here are significant in that many questions which were previously answerable only qualitatively or intuitively should now be amenable to quantitative study. The relationships presented can be used to evaluate realistically propagation paths where the statistics of the medium are slowly varying, e.g., communications through the atmosphere from space to earth, earth to space, and earth to earth.

2. Wave propagation in a constant medium. Before illustrating the techniques used to analyze wave propagation in more general media, we shall consider the nature of a focused beam of light propagated in a constant medium, and thereby obtain an equation for the wave function of an unperturbed beam of light. This formulation will serve subsequently to demonstrate the application of our results to a specific example of determining the statistical effects of randomness of the transmission medium on a prescribed, unperturbed wave function.

For this purpose, let $u_0(\mathbf{r})$ represent an unperturbed wave function, i.e., propagated in a constant medium; $u_0(\mathbf{r})$ may, for example, be described as a collimated beam of light such as would be emitted by a focused transmitting laser. The position vector $\mathbf{r} = \mathbf{z} + \boldsymbol{\rho}$ is located in a cylindrical coordinate system, with $\mathbf{z}$, the axial coordinate vector, and $\boldsymbol{\rho}$, the radial coordinate vector. The wave function $u_0(\mathbf{r})$ is chosen to represent the propagation of light due to a disturbance

$$u_0(\mathbf{r})]_{z=0} = u_0(\boldsymbol{\rho}) = \exp\left[-\frac{\rho^2}{2\alpha_0^2} + \frac{ik\rho^2}{2R}\right] \tag{2.1}$$

located at the exit pupil plane $z = 0$, and is assumed to have a Gaussian amplitude distribution and a focused wavefront. Here α_0 is a parameter measuring the effective beam radius, and R, the radius of curvature of the focused wavefront of the beam, produced say, by the action of an ideal lens; $\rho = |\boldsymbol{\rho}|$. This particular choice has considerable significance when applied to the study of the limitations of communication by transmission of coherent light through the atmosphere. The initial disturbance represented by $u_0(\boldsymbol{\rho})$ causes a wave $u_0(\mathbf{r})$ to propagate through the atmosphere in the space $z > 0$, where it satisfies the reduced wave equation

$$\nabla^2 u_0(\mathbf{r}) + k^2 u(\mathbf{r}) = 0, \qquad z > 0. \tag{2.2}$$

Throughout this work, we shall take for granted the monochromaticity of our solution and suppress the harmonic time factor exp $(i\omega t)$. The refractive index is taken as unity and k is $2\pi/\lambda$, the optical wave number.

To solve (2.2) subject to the boundary condition (2.1), we apply Green's theorem to the interior region, $z > 0$, bounded by a large hemisphere and the plane $z = 0$. The integral over the hemisphere can be made arbitrarily small and we obtain

$$u_0(\mathbf{r}) = \int d\boldsymbol{\rho}_0\, u_0(\boldsymbol{\rho}_0) \frac{\partial}{\partial z_0} G_0(\mathbf{r};\, \boldsymbol{\rho}_0) \tag{2.3}$$

where the integral extends over the entire plane $z = 0$, and the Green's function $G_0(\mathbf{r};\, \mathbf{r}_0)$

vanishes on $z = 0$, and satisfies

$$\nabla^2 G_0 + k^2 G_0 = -\delta(\mathbf{r} - \mathbf{r}_0), \qquad z > 0. \tag{2.4}$$

Hence

$$G_0(\mathbf{r}, \mathbf{r}_0) = G(|\mathbf{r} - \mathbf{r}_0|) - G(|\mathbf{r} - \mathbf{r}_0'|) \tag{2.5}$$

where

$$G(r) = \frac{1}{4\pi r} \exp ikr \tag{2.6}$$

and

$$|\mathbf{r} - \mathbf{r}_0| = [(z - z_0)^2 + (\boldsymbol{\varrho} - \boldsymbol{\varrho}_0)^2]^{1/2},$$
$$|\mathbf{r} - \mathbf{r}_0'| = [(z + z_0)^2 + (\boldsymbol{\varrho} - \boldsymbol{\varrho}_0)^2]^{1/2}.$$

Since

$$\frac{\partial}{\partial z_0} G_0(\mathbf{r}, \boldsymbol{\varrho}_0) = \lim_{z_0 \to 0} \frac{\partial}{\partial z_0} [G(|\mathbf{r} - \mathbf{r}_0|) - G(|\mathbf{r} - \mathbf{r}_0'|)]$$
$$= -2 \frac{\partial}{\partial z} G(|\mathbf{r} - \boldsymbol{\varrho}_0|),$$

we have

$$u_0(\mathbf{r}) = -2 \frac{\partial}{\partial z} \int d\boldsymbol{\varrho}_0 \, u_0(\boldsymbol{\varrho}_0) G(|\mathbf{r} - \boldsymbol{\varrho}_0|) \tag{2.7}$$

where G is given by (2.6) and u_0 is chosen as

$$u_0(\boldsymbol{\varrho}_0) = \exp(-\rho_0^2/2\alpha^2) \tag{2.8}$$

with

$$1/\alpha^2 = 1/\alpha_0^2 - ik/R. \tag{2.9}$$

Since we are interested in the wave function in the region centered about the principal ray, one can expand

$$|\mathbf{r} - \boldsymbol{\varrho}_0| \sim z + \frac{|\boldsymbol{\varrho} - \boldsymbol{\varrho}_0|^2}{2z} + \cdots \tag{2.10}$$

and neglect higher order terms. It is perfectly reasonable to make the approximation for visible optics that $k \gg 1/z$. Under these conditions, (2.7) yields

$$u_0(\mathbf{r}) \sim \frac{ke^{ikz}}{2\pi iz} \int d\boldsymbol{\varrho}_0 \exp\left[-\frac{\rho_0^2}{2\alpha^2} + \frac{ik}{2z} |\boldsymbol{\varrho} - \boldsymbol{\varrho}_0|^2\right] \tag{2.11}$$

which evaluated becomes

$$u_0(\mathbf{r}) \sim -\frac{ik\alpha^2}{Z} \exp(-k^2\alpha^2) \exp[ik(Z + \rho^2/2Z)], \qquad z \gg \alpha_0 \gg 1/k \tag{2.12}$$

where

$$Z = z - ik\alpha^2. \tag{2.13}$$

Eq. (2.12) represents an optical approximation for the Gaussian beam of focused light as it propagates through a medium with an index of refraction equal to unity. This specific result will be applied later in Sec. 5 where computations apply to the Gaussian light source.

3. Wave propagation in a non-constant medium. Wave propagation in a non-constant, continuous medium is governed by the wave equation

$$\nabla^2 u(\mathbf{r}) + k^2 n^2(\mathbf{r}) u(\mathbf{r}) = 0 \tag{3.1}$$

where $n(\mathbf{r})$ is the refractive index function characterizing the transmission medium. Applying the Rytov transformation, $u = \exp \Psi$, takes (3.1) into the equivalent non-linear spatial Riccati equation

$$\nabla^2 \Psi + (\nabla \Psi)^2 = -k^2 n^2(\mathbf{r}) \tag{3.2}$$

where

$$\Psi = \log_e A(\mathbf{r}) + i\phi(\mathbf{r}), \tag{3.3}$$

$A(\mathbf{r})$ and $\phi(\mathbf{r})$ being the amplitude and phase associated with $u(\mathbf{r})$ as

$$u(\mathbf{r}) = A(\mathbf{r}) \exp i\phi(\mathbf{r}). \tag{3.4}$$

Now $n(\mathbf{r})$ may be written as

$$n(\mathbf{r}) = 1 + \epsilon\mu(\mathbf{r}) \tag{3.5}$$

where the parameter ϵ has been inserted to measure the deviation of the refractive index from its mean value unity.

It seems reasonable to suppose that Ψ can be represented in a power series in ϵ

$$\Psi(\mathbf{r}; \epsilon) = \psi_0(\mathbf{r}) + \epsilon\psi_1(\mathbf{r}) + \epsilon^2\psi_2(\mathbf{r}) + \cdots . \tag{3.6}$$

Substituting (3.6) into (3.2) then gives

$$\sum_{m=0}^{\infty} \epsilon^m \left[\nabla^2 \psi_m + \sum_{p=0}^{m} \nabla\psi_p \cdot \nabla\psi_{m-p} \right] = -k^2 - \epsilon 2k^2\mu - \epsilon^2 k^2 \mu^2. \tag{3.7}$$

Equating to zero the coefficient of each power of ϵ, we obtain

$$\nabla^2\psi_0 + \nabla\psi_0 \cdot \nabla\psi_0 = -k^2, \tag{3.8}$$

$$\nabla^2\psi_1 + 2\nabla\psi_0 \cdot \nabla\psi_1 = -2k^2\mu, \tag{3.9}$$

$$\nabla^2\psi_2 + 2\nabla\psi_0 \cdot \nabla\psi_2 = -k^2\mu^2 - \nabla\psi_1 \cdot \nabla\psi_1 , \tag{3.10}$$

$$\nabla^2\psi_m + 2\nabla\psi_0 \cdot \nabla\psi_m = -\sum_{p=1}^{m-1} \nabla\psi_p \cdot \nabla\psi_{m-p} , \qquad m = 3, 4, 5, \cdots . \tag{3.11}$$

Eq. (3.8) gives the solution of (3.2) for the case $\epsilon = 0$. Thus, if $u_0 = \exp \psi_0$, then (3.8) is equivalent to

$$\nabla^2 u_0 + k^2 u_0 = 0. \tag{3.12}$$

Equations (3.9)–(3.11) are a system which can be solved successively for the ψ_m's. To accomplish this, we introduce the functions

$$W_m = u_0 \psi_m , \qquad m = 1, 2, 3, \cdots . \tag{3.13}$$

Using (3.8) we find

$$\begin{aligned}\nabla^2 W_m &= \nabla^2(u_0\psi_m)\\ &= u_0[\nabla^2\psi_m + 2\nabla\psi_0\nabla\psi_m + \psi_m[\nabla^2\psi_0 + \nabla\psi_0\cdot\nabla\psi_0]]\\ &= u_0[\nabla^2\psi_m + 2\nabla\psi_0\cdot\nabla\psi_m] - k^2 W_m\end{aligned}$$

so that (3.9)–(3.11) can be written as

$$\nabla^2 W_1 + k^2 W_1 = -2k^2\mu u_0 \ , \tag{3.14}$$

$$\nabla^2 W_2 + k^2 W_2 = -[k^2\mu^2 + \nabla(W_1/u_0)\cdot\nabla(W_1/u_0)]u_0 \ , \tag{3.15}$$

$$\nabla^2 W_m + k^2 W_m = -\sum_{p=1}^{m-1} [\nabla(W_p/u_0)\cdot\nabla(W_{m-p}/u_0)u_0], \qquad m = 3, 4, 5, \cdots . \tag{3.16}$$

The solution of equation set (3.14)–(3.16) are given iteratively by

$$W_1(\mathbf{r}) = 2k^2 \int \mu(\mathbf{r}_0)u_0(\mathbf{r}_0)G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 \ , \tag{3.17}$$

$$W_2(\mathbf{r}) = \int [k^2\mu^2 + \nabla(W_1/u_0)\cdot\nabla(W_1/u_0)]u_0(\mathbf{r}_0)G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 \ , \tag{3.18}$$

$$W_m(\mathbf{r}) = \sum_{p=1}^{m-1} \int \nabla(W_p/u_0)\cdot\nabla(W_{m-p}/u_0)u_0(\mathbf{r}_0)G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 \ , \quad m = 3, 4, 5, \cdots \tag{3.19}$$

where the integration is taken over that part of space from which scattered waves arrive at the point **r**. From (3.13)

$$\psi_1(\mathbf{r}) = 2k^2 \int \mu(\mathbf{r}_0)\, \frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})}\, G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 \ , \tag{3.20}$$

$$\psi_2(\mathbf{r}) = \int [k^2\mu^2 + \nabla\psi_1\cdot\nabla\psi_1]\, \frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})}\, G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 \ , \tag{3.21}$$

$$\psi_m(\mathbf{r}) = \sum_{p=1}^{m-1} \int [\nabla\psi_p\cdot\nabla\psi_{m-p}]\, \frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})}\, G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 \ , \qquad m = 3, 4, 5, \cdots . \tag{3.22}$$

Eqs. (3.20)–(3.22) together with the solution of (3.12) represent the general solution for wave propagation in a nonconstant, continuous medium.

The solution of (3.1) can therefore be written in the form

$$u(\mathbf{r}) = u_0(\mathbf{r}) \exp \psi(\mathbf{r}) \tag{3.23}$$

where $u_0(\mathbf{r})$ may be thought of as an unperturbed wave function propagating as if the transmission medium were constant, and where

$$\psi = \Psi - \psi_0 = \log_e \left|\frac{A(\mathbf{r})}{A_0(\mathbf{r})}\right| + i[\phi(\mathbf{r}) - \phi_0(\mathbf{r})] \tag{3.24}$$

is a measure of the deviation of the logarithm of the amplitude and phase from their unperturbed values, A_0 and ϕ_0 , respectively. The function ψ is computed from

$$\psi(\mathbf{r}) = \epsilon\psi_1(\mathbf{r}) + \epsilon^2\psi_2(\mathbf{r}) + \epsilon^3\psi_3(\mathbf{r}) + \cdots \tag{3.25}$$

where the ψ_m's are given by (3.20)–(3.22).

4. Wave propagation in a random continuous medium. Let us consider next what happens when the beam propagates through an inhomogeneous medium having a randomly fluctuating index of refraction. In this case, the beam of light is bent and the wavefront is distorted as the beam propagates until eventually, at sufficiently large distances, the coherence of the wave is completely destroyed.

Before proceeding, let us recall what is meant by wave propagation in a random continuous medium. A random medium is characterized by a family of media having a family of index of refraction functions, $n(\mathbf{r};\, \omega)$, depending on a position vector $\mathbf{r}$, and a parameter ω, identifying a particular member medium of the family, together with a probability distribution specifying the probability of various members of the family. Wave propagation in a random medium therefore refers to propagation in each member medium together with the probability of each member medium. This probability, when associated with wave motion in each medium, characterizes a random wave motion.

Wave propagation in random continuous media can be studied by considering the solution wave function $u(\mathbf{r};\, \omega)$ for each member medium with a refractive index $n(\mathbf{r};\, \omega)$. Each member solution is governed by a stochastic differential equation of the form

$$\nabla^2 u(\mathbf{r};\omega) + k^2 n^2(\mathbf{r};\omega) u(\mathbf{r};\omega) = 0. \tag{4.1}$$

Here the coefficient $n(\mathbf{r};\, \omega)$ characterizes the random propagation medium, as a family of index of refraction functions. If the problem is well posed, then $n(\mathbf{r};\, \omega)$ is measurable with respect to a probability measure $dP(\omega)$ and there exists a solution, $u(\mathbf{r};\, \omega)$ for each point ω in probability space. This family of solutions, with the probability measure $dP(\omega)$, for each value of ω, then characterizes the wave propagation in a continuous random medium.

To study the statistics of wave propagation in random media, definitions of certain statistical properties of random functions are needed. The mean value of a random function $f(\mathbf{r};\, \omega)$, is defined as the ensemble average of all possible values the function f can assume at the point $\mathbf{r}$, weighted by a probability measure $dP(\omega)$. The mean value of $f(\mathbf{r};\, \omega)$, denoted $\langle f(\mathbf{r}) \rangle$, is defined by

$$\langle f(\mathbf{r}) \rangle = \int_{-\infty}^{\infty} f(\mathbf{r};\omega)\, dP(\omega). \tag{4.2}$$

The covariance of the random function $f(r;\, \omega)$ is then

$$C_f(\mathbf{r}_1\, ;\mathbf{r}_2) = \langle |f(\mathbf{r}_1) - \langle f(\mathbf{r}_1)\rangle| \cdot |f(\mathbf{r}_2) - \langle f(\mathbf{r}_2)\rangle| \rangle \tag{4.3}$$

which should be distinguished from the correlation function

$$B_f(\mathbf{r}_1\, ;\mathbf{r}_2) = \langle |f(\mathbf{r}_1) f(\mathbf{r}_2)| \rangle. \tag{4.4}$$

If the mean value of f is set equal to zero, only then can the covariance function and the correlation function be identified as equal.*

A random function is said to be globally stationary if its statistical characteristics are invariant under translation, i.e., if they remain the same when $\mathbf{r}$ is replaced by $\mathbf{r} + \mathbf{a}$,

*It should be noted that Tatarski avoids any distinction between the two functions and refers to both as correlation functions [1]. The distinction is unnecessary in his work, however, since he treats the mean values of phase and log-amplitude fluctuations about their unperturbed values as zero, a result which shall be shown to be not the case.

where **a** is an arbitrary displacement vector. In particular, it follows that the mean value of a stationary function f must be a constant, which is denoted

$$\langle f(\mathbf{r})\rangle = \langle f\rangle. \tag{4.5}$$

Similarly, the covariance function for a globally stationary, isotropic function $f(\mathbf{r};\omega)$ depends only on the distance between the two points being correlated, i.e.,

$$C_f(\mathbf{r}_1\,;\mathbf{r}_2) = C_f(|\mathbf{r}_1 - \mathbf{r}_2|) \tag{4.6}$$

$$B_f(\mathbf{r}_1\,;\mathbf{r}_2) = B_f(|\mathbf{r}_1 - \mathbf{r}_2|). \tag{4.7}$$

The mean square continuity of a random function means

$$\lim_{t\to 0}\langle |f(\mathbf{r}+\mathbf{t}) - f(\mathbf{r})|^2\rangle = 0. \tag{4.8}$$

The function

$$D_f(\mathbf{t}) = \langle |f(\mathbf{r}+\mathbf{t}) - f(\mathbf{r})|^2\rangle \tag{4.9}$$

is called the structure function and is the basic characteristic of a random process with stationary increments [1]. In a sense, the value of $D_f(\mathbf{t})$ characterizes the intensity of those fluctuations of f with periods which are smaller than or comparable with t. The connection between the structure function D_f and the covariance function C_f for a stationary random function f can be made clear by expanding (4.9)

$$\begin{aligned} D_f(\mathbf{t}) &= \langle |f(\mathbf{r}+\mathbf{t}) - f(\mathbf{r})|\cdot|f(\mathbf{r}+\mathbf{t}) - f(\mathbf{r})|\rangle \\ &= 2[C_f(0) - C_f(\mathbf{t})]. \end{aligned} \tag{4.10}$$

It is interesting to observe that from this result it follows that the continuity of the covariance function at $t = 0$ in the ordinary sense implies the continuity of f in the mean square sense. Thus f must have finite variance if the mean square properties are to hold. In the case where $C_f(\infty) = 0$, which is usually the case, we have

$$C_f(\mathbf{r}) = \tfrac{1}{2}[D_f(\infty) - D_f(\mathbf{r})]. \tag{4.11}$$

In the ensuing analyses, it is assumed that the mean value of the index of refraction $\langle n(\mathbf{r})\rangle$ is unity. The covariance function for the refractive index is accordingly

$$C_n(|\mathbf{r}_1 - \mathbf{r}_2|) = \langle [n(\mathbf{r}_1) - 1][n(\mathbf{r}_2) - 1]\rangle. \tag{4.12}$$

Here we have expressed C_n as a function of the distance $|\mathbf{r}_1 - \mathbf{r}_2|$ between the two points being correlated, thus implying global stationariness. This is an appropriate notation when the transmission medium is both homogeneous and isotropic in space.

We shall also need to know the covariance of the gradient of the index of refraction, $\nabla n(\mathbf{r};\omega)$. This can be expressed in terms of the covariance function C_n by the following limiting process

$$\begin{aligned} C_{\nabla n}(|\mathbf{r}_1 - \mathbf{r}_2|) &= \langle \nabla n(\mathbf{r}_1)\cdot\nabla n(\mathbf{r}_2)\rangle, \\ &= \lim_{t,t'\to 0}\langle \nabla_t n(\mathbf{r}_1+\mathbf{t})\cdot\nabla_{t'} n(\mathbf{t}'+\mathbf{r}_2)\rangle, \\ &= \lim_{t,t'\to 0}\nabla_t\cdot\nabla_{t'}\langle [n(\mathbf{r}_1+\mathbf{t}) - 1][n(\mathbf{r}_2+\mathbf{t}') - 1]\rangle, \\ &= \lim_{t,t'\to 0}\nabla_t\cdot\nabla_{t'} C_n(|\mathbf{r}_1 + \mathbf{t} - \mathbf{r}_2 - \mathbf{t}'|), \\ &= -\nabla^2 C_n(|\mathbf{r}_1 - \mathbf{r}_2|). \end{aligned} \tag{4.13}$$

We next define what is meant by a locally stationary process as contrasted to one which is globally stationary (over all space) [5]. The concept has considerably more physical significance here since the atmosphere is known to be only homogeneous locally. The statistics vary smoothly and slowly as a function of position along the path of propagation, e.g., as a function of altitude. In this paper, a process shall be considered to be stationary, isotropic locally if the covariance functions can be put into the form

$$C_f(\mathbf{r}_1 ; \mathbf{r}_2) = C_f(|\mathbf{r}_1 - \mathbf{r}_2|; s), \tag{4.14}$$

$$C_{\nabla f}(\mathbf{r}_1 ; \mathbf{r}_2) = -\nabla^2 C_f(|\mathbf{r}_1 - \mathbf{r}_2|; s) \tag{4.15}$$

where s is the distance $\frac{1}{2} |\mathbf{r}_1 + \mathbf{r}_2|$; and the covariance functions are invariant under translation but weakly dependent on s, a parameter which determines the local behavior. The index of refraction is assumed to be locally stationary, and isotropic and therefore (4.14) and (4.15) are appropriate representations for its covariance functions.

5. Mean value of phase and log-amplitude. The mean value of Ψ, denoted by $\langle\Psi\rangle$ is found by taking the ensemble average of each term in (3.6)

$$\langle\Psi(r)\rangle = \langle\psi_0(\mathbf{r})\rangle + \epsilon\langle\psi_1(\mathbf{r})\rangle + \epsilon^2\langle\psi_2(\mathbf{r})\rangle + \cdots . \tag{5.1}$$

Since $\psi_0(\mathbf{r})$ is nonrandom, $\langle\psi_0(\mathbf{r})\rangle = \psi_0(\mathbf{r})$. The ensemble average of $\psi_1(\mathbf{r})$ may be computed from (3.20) by interchanging the order of integrating and averaging, which is assumed permissible. Then

$$\langle\psi_1(\mathbf{r})\rangle = 2k^2 \int \langle\mu(\mathbf{r}_0)\rangle \frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})} G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 . \tag{5.2}$$

Since $\langle\mu(\mathbf{r}_0)\rangle = 0$, we have $\langle\psi_1(\mathbf{r})\rangle = 0$. From (3.21) we find

$$\langle\psi_2(\mathbf{r})\rangle = \int [k^2\langle\mu^2(\mathbf{r}_0)\rangle + \langle\nabla\psi_1(\mathbf{r}_0)\cdot\nabla\psi_1(\mathbf{r}_0)\rangle] \frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})} G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 . \tag{5.3}$$

Therefore, the mean value of Ψ is given by

$$\langle\Psi(\mathbf{r})\rangle = \psi_0(\mathbf{r}) + \epsilon^2 k^2 \int \langle\mu^2(\mathbf{r}_0)\rangle \frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})} G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 + M(\mathbf{r}) + O(\epsilon^3) \tag{5.4}$$

where

$$M(\mathbf{r}) = \epsilon^2 \int \langle\nabla\psi_1(\mathbf{r}_0)\cdot\nabla\psi_1(\mathbf{r}_0)\rangle \frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})} G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 \tag{5.5}$$

and ψ_1 is given by (3.20).

Multiplying (3.20) by $u_0(\mathbf{r})$ and taking the gradient of the product gives

$$\begin{aligned} \nabla[\psi_1(\mathbf{r})u_0(\mathbf{r})] &= 2k^2 \int \mu(\mathbf{r}_0)u_0(\mathbf{r}_0)\nabla_{\mathbf{r}}G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 , \\ &= -2k^2 \int \mu(\mathbf{r}_0)u_0(\mathbf{r}_0)\nabla_{\mathbf{r}_0}G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 . \end{aligned} \tag{5.6}$$

Integration by parts leads to

$$\nabla[\psi_1(\mathbf{r})u_0(\mathbf{r})] = 2k^2 \int \nabla[\mu(\mathbf{r}_0)u_0(\mathbf{r}_0)]G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 \,. \tag{5.7}$$

Next we use the identity

$$\nabla\psi_1(\mathbf{r}) = \frac{\nabla[\psi_1(\mathbf{r})u_0(\mathbf{r})]}{u_0(\mathbf{r})} - \psi_1(\mathbf{r})\nabla\psi_0(\mathbf{r}) \tag{5.8}$$

and (3.20) to obtain

$$\begin{aligned}\nabla\psi_1(\mathbf{r}) &= 2k^2 \int \left[\frac{\nabla[\mu(\mathbf{r}_0)u_0(\mathbf{r}_0)]}{u_0(\mathbf{r}_0)} - \mu(\mathbf{r}_0)\nabla\psi_0(\mathbf{r})\right] \frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})}\, G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 \\ &= 2k^2 \int \{\nabla\mu(\mathbf{r}_0) - \mu(\mathbf{r}_0)[\nabla\psi_0(\mathbf{r}) - \nabla\psi_0(\mathbf{r}_0)]\} \frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})}\, G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 \,.\end{aligned} \tag{5.9}$$

It is perfectly permissible to make the assumption for visible optics that $k = 2\pi/\lambda \gg 1/r_0$, and

$$\left|\frac{\nabla\mu(\mathbf{r}_0)}{\mu(\mathbf{r}_0)}\right| \gg |\nabla\psi_0(\mathbf{r}) - \nabla\psi_0(\mathbf{r}_0)|. \tag{5.10}$$

Hence (5.9) becomes

$$\nabla\psi_1(\mathbf{r}) \cong 2k^2 \int \nabla\mu(\mathbf{r}_0)\, \frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})}\, G(|\mathbf{r} - \mathbf{r}_0|)\, d\mathbf{r}_0 \,. \tag{5.11}$$

The wave function u_0 is chosen to represent the unperturbed wave function of the focused laser beam given by (2.12). Thus

$$\frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})} = \frac{Z}{Z_0} \exp ik\left\{z_0 - z + \frac{1}{2}\left[\frac{\rho_0^2}{Z_0} - \frac{\rho^2}{Z}\right]\right\}. \tag{5.12}$$

It seems reasonable that only those inhomogenieties which are located within a cone with vertex at $\mathbf{r}$ with a small aperture should contribute significantly to the evaluation of (5.11). Inside this cone,

$$\begin{aligned}G(|\mathbf{r} - \mathbf{r}_0|) &= \frac{\exp ik\, |\mathbf{r} - \mathbf{r}_0|}{4\pi\, |\mathbf{r} - \mathbf{r}_0|} \\ &\sim \frac{\exp ik\, |z - z_0|}{4\pi\, |z - z_0|} \exp\left\{\frac{ik}{2} \frac{|\boldsymbol{\rho} - \boldsymbol{\rho}_0|^2}{|z - z_0|}\right\}.\end{aligned} \tag{5.13}$$

Reflections should also play a minor role and we shall consequently limit the integration to the region $0 \le z_0 \le z$. Therefore, substituting (5.12) and (5.13) into (5.11), we find that $\nabla\psi_1$ can be represented by a simpler expression

$$\nabla\psi_1(\mathbf{r}) = -\frac{ik}{\pi} \int_0^z \frac{dz_0}{\gamma(z, z_0)} \int d\boldsymbol{\rho}_0\, \nabla\mu(\mathbf{r}_0) \exp\left[\frac{(\boldsymbol{\rho}_0 - \boldsymbol{\rho}Z_0/Z)^2}{\gamma(z, z_0)}\right] \tag{5.14}$$

where the $d\boldsymbol{\rho}_0$ integration is taken over the entire two dimensional vector space of $\boldsymbol{\rho}_0$ and

$$\gamma(z, z_0) = \frac{2(z - z_0)Z_0}{ikZ}. \tag{5.15}$$

Consequently,

$$\begin{aligned}\langle \nabla \psi_1(\mathbf{r}_0) \cdot \nabla \psi_1(\mathbf{r}_0) \rangle \\ &= -\left(\frac{k}{\pi}\right)^2 \int_0^{z_0} \int_0^{z_0} \frac{dz_1\, dz_2}{\gamma(z_0\,, z_1)\gamma(z_0\,, z_2)} \cdot \int d\boldsymbol{\varrho}_1 \int d\boldsymbol{\varrho}_2 \, \langle \nabla \mu(\mathbf{r}_1) \cdot \nabla \mu(\mathbf{r}_2) \rangle \\ &\quad \cdot \exp\left[\frac{(\boldsymbol{\varrho}_1 - \boldsymbol{\varrho}_0 Z_1/Z_0)^2}{\gamma(z_0\,, z_1)} + \frac{(\boldsymbol{\varrho}_2 - \boldsymbol{\varrho}_0 Z_2/Z_0)^2}{\gamma(z_0\,, z_2)}\right]. \end{aligned} \tag{5.16}$$

Now from (4.15)

$$\epsilon^2 \langle \nabla \mu(\mathbf{r}_1) \cdot \nabla \mu(\mathbf{r}_2) \rangle = -\nabla^2 C_n(|\mathbf{r}_1 - \mathbf{r}_2|; \tfrac{1}{2}(z_1 + z_2)) \tag{5.17}$$

which is appropriate here since the refractive index of the medium can be considered locally stationary. Substituting (5.17) into (5.16) and introducing the change of variables

$$\boldsymbol{\zeta} = \boldsymbol{\varrho}_1 - \boldsymbol{\varrho}_2\,, \qquad \mathbf{n} = \tfrac{1}{2}(\boldsymbol{\varrho}_1 + \boldsymbol{\varrho}_2), \tag{5.18}$$

yields

$$\begin{aligned}\epsilon^2 \langle \nabla \psi_1(\mathbf{r}_0) \cdot \nabla \psi_1(\mathbf{r}_0) \rangle \\ &= \left(\frac{k}{\pi}\right)^2 \int_0^{z_0} \int_0^{z_0} \frac{dz_1\, dz_2}{\gamma(z_0\,, z_1)\gamma(z_0\,, z_2)} \int d\boldsymbol{\zeta} \, \nabla^2 C_n[|(z_1 - z_2)^2 + \zeta^2|^{1/2}; \tfrac{1}{2}(z_1 + z_2)] \\ &\quad \cdot \int d\mathbf{n} \, \exp\left[\frac{(\mathbf{n} + \mathbf{a})^2}{\gamma(z_0\,, z_1)} + \frac{(\mathbf{n} + \mathbf{b})^2}{\gamma(z_0\,, z_2)}\right] \end{aligned} \tag{5.19}$$

where

$$\begin{aligned} \mathbf{a} &= \tfrac{1}{2}\boldsymbol{\zeta} - \boldsymbol{\varrho}_0 Z_1/Z_0\,, \\ \mathbf{b} &= -\tfrac{1}{2}\boldsymbol{\zeta} - \boldsymbol{\varrho}_0 Z_2/Z_0\,. \end{aligned} \tag{5.20}$$

The $d\mathbf{n}$ integration can be performed explicitly by substituting the result

$$\int \exp\left[\frac{(\mathbf{n} + \mathbf{a})^2}{\gamma_a} + \frac{(\mathbf{n} + \mathbf{b})^2}{\gamma_b}\right] d\mathbf{n} = -\pi\left[\frac{\gamma_a \gamma_b}{\gamma_a + \gamma_b}\right] \exp\left[\frac{(\mathbf{a} - \mathbf{b})^2}{\gamma_a + \gamma_b}\right]. \tag{5.21}$$

Thus (5.19) becomes

$$\begin{aligned}\epsilon^2 \langle \nabla \psi_1(\mathbf{r}_0) \cdot \nabla \psi_1(\mathbf{r}_0) \rangle &= -\frac{k^2}{\pi} \int_0^{z_0} \int_0^{z_0} \frac{dz_1\, dz_2}{\gamma(z_0\,, z_1) + \gamma(z_0\,, z_2)} \\ &\quad \cdot \int d\boldsymbol{\zeta} \, \nabla^2 C_n[|(z_1 - z_2)^2 + \zeta^2|^{1/2}; \tfrac{1}{2}(z_1 + z_2)] \exp\left[\frac{(\boldsymbol{\zeta} - \boldsymbol{\varrho}_0(z_1 - z_2)/Z_0)^2}{\gamma(z_0\,, z_1) + \gamma(z_0\,, z_2)}\right]. \end{aligned} \tag{5.22}$$

The approximations for visible optics (5.12), (5.13), when applied to the integral representing $M(\mathbf{r})$ of (5.5) yields

$$M(\mathbf{r}) = \frac{\epsilon^2}{2\pi i k} \int_0^{z} \frac{dz_0}{\gamma(z, z_0)} \int d\boldsymbol{\varrho}_0 \, \langle \nabla \psi_1(\mathbf{r}_0) \cdot \nabla \psi_1(\mathbf{r}_0) \rangle \exp\left[\frac{(\boldsymbol{\varrho}_0 - \boldsymbol{\varrho} Z_0/Z)^2}{\gamma(z, z_0)}\right]. \tag{5.23}$$

Upon substituting (5.22) into (5.23) we obtain

$$\begin{aligned} M(\mathbf{r}) &= \frac{ik}{2\pi^2} \int_0^{z} \frac{dz_0}{\gamma(z, z_0)} \int_0^{z_0} \int_0^{z_0} \frac{dz_1\, dz_2}{\gamma(z_0\,, z_1) + \gamma(z_0\,, z_2)} \\ &\quad \cdot \int d\boldsymbol{\zeta} \, \nabla^2 C_n[|(z_1 - z_2)^2 + \zeta^2|^{1/2}; \tfrac{1}{2}(z_1 + z_2)] \\ &\quad \cdot \int d\boldsymbol{\varrho}_0 \, \exp\left[\frac{(\boldsymbol{\varrho}_0 - \boldsymbol{\varrho} Z_0/Z)^2}{\gamma(z, z_0)} + \frac{(\boldsymbol{\varrho}_0 - \boldsymbol{\zeta} Z_0/(z_1 - z_2))^2}{[Z_0/(z_1 - z_2)]^2[\gamma(z_0\,, z_1) + \gamma(z_0\,, z_2)]}\right]. \end{aligned} \tag{5.24}$$

The $d\varrho_0$ integration is performed using (5.21):

$$M(\mathbf{r}) = -\frac{ik}{2\pi}\int_0^z dz_0 \int_0^{z_0}\int_0^{z_0} dz_1\, dz_2 \int \frac{d\zeta}{\Lambda}\, \nabla^2 C_n[|(z_1 - z_2)^2 + \zeta^2|^{1/2}; s] \cdot \exp\left[\frac{(\zeta - \varrho(z_1 - z_2)/Z)^2}{\Lambda}\right] \tag{5.25}$$

where

$$\Lambda = \gamma(z_0, z_1) + \gamma(z_0, z_2) + [(z_1 - z_2)/Z_0]^2\gamma(z, z_0), \tag{5.26}$$

$$s = \tfrac{1}{2}(z_1 + z_2). \tag{5.27}$$

We next introduce the change of variables $t = z_1 - z_2$ into Eq. (5.25):

$$M(\mathbf{r}) = -\frac{ik}{2\pi}\int_0^z dz_0 \int_0^{z_0} ds \int_{-z_0+2|s-(1/2)z_0|}^{z_0-2|s-(1/2)z_0|} dt \int \frac{d\zeta}{\Lambda}\, \nabla^2 C_n[|t^2 + \zeta^2|^{1/2}; s] \cdot \exp\left[\frac{(\zeta - \varrho t/Z)^2}{\Lambda}\right] \tag{5.28}$$

and into (5.26):

$$\Lambda = 2\gamma(z_0, s) - t^2/ikZ. \tag{5.29}$$

The covariance function $C_n(\zeta)$ falls off rapidly to zero for points separated by a distance $\zeta > a_c$ which we call a correlation distance. Now since a_c is a relatively small distance compared to path length, there are several approximations that can be made which will simplify the evaluation of (5.28) considerably. Since

$$|z_0 - 2\,|s - \tfrac{1}{2}z_0|| \gg a_c \geq t \tag{5.30}$$

for values of s larger than a few times a_c or values of $|z_0 - s|$ larger than a few times a_c (the distances from the end points, $z = 0$, and $z = z_0$), the limits in the dt integral in (5.28) can be extended between $-\infty$ and $+\infty$ with negligible error.

Likewise, the coefficient Λ of (5.29) has a very weak dependence on t in the range for which t is not so large as to make C_n vanish. We shall therefore set $t = 0$ in (5.29). Consequently, (5.28) can be expressed more simply as

$$M(\mathbf{r}) = -\frac{ik}{4\pi}\int_0^z dz_0 \int_0^{z_0} \frac{ds}{\gamma(z_0, s)} \int_{-\infty}^{+\infty} dt \int d\zeta\, \nabla^2 C_n[|t^2 + \zeta^2|^{1/2}; s] \cdot \exp\left[\frac{(\zeta - \varrho t/Z)^2}{2\gamma(z_0, s)}\right]. \tag{5.31}$$

Examination of the relative magnitudes of the terms in the exponential in (5.31) shows that for $|\varrho/Z| \ll 1$, the dependence on ϱ is quite weak. For this reason, we shall set $\varrho = 0$ and ignore this dependence entirely. This also serves to eliminate the dependence of the exponential on t. Consequently, the dt integration can be carried out assuming C_n is prescribed. Thus we shall compute the on axis value of M as given by

$$M(z) = -ik \int_0^z dz_0 \int_0^{z_0} \frac{ds}{\gamma(z_0, s)} \int_0^\infty \zeta f_0(\zeta, s) \exp\left[\frac{\zeta^2}{2\gamma(z_0, s)}\right] d\zeta \tag{5.32}$$

where

$$f_0(\zeta, s) = \int_0^\infty \nabla^2 C_n[|t^2 + \zeta^2|^{1/2}; s]\, dt. \tag{5.33}$$

Proceeding with the evaluation of (5.4) we note that the variance of the index of refraction is given by

$$\epsilon^2\langle\mu^2(\mathbf{r}_0)\rangle = C_n(0, z_0). \tag{5.34}$$

To evaluate the volume integral in (5.4), we use this result together with (5.12) and (5.13)

$$\begin{aligned}\epsilon^2 \frac{k^2}{4\pi}\int \langle\mu^2\rangle \frac{u_0(\mathbf{r}_0)}{u_0(\mathbf{r})} \frac{\exp ik\,|\mathbf{r}-\mathbf{r}_0|}{|\mathbf{r}-\mathbf{r}_0|}\, d\mathbf{r}_0 \\ = -\frac{ik}{2\pi}\int_0^z \frac{C_n(0, z_0)}{\gamma(z, z_0)}\, dz_0 \int d\boldsymbol{\varrho}_0 \exp\left[\frac{(\boldsymbol{\varrho}_0 - \boldsymbol{\varrho}Z_0/Z)^2}{\gamma(z, z_0)}\right] \\ = \frac{ik}{2}\int_0^z C_n(0; s)\, ds.\end{aligned} \tag{5.35}$$

Therefore the mean value of the fluctuation function ψ is given by

$$\begin{aligned}\langle\psi(z)\rangle = \frac{ik}{2}\int_0^z C_n(0, z_0)\, dz_0 \\ - ik\int_0^z dz_0 \int_0^{z_0} \frac{ds}{\gamma(z_0\,, s)} \int_0^\infty \zeta f_0(\zeta, s) \exp\left[\frac{\zeta^2}{2\gamma(z_0\,, s)}\right] d\zeta + O(\epsilon^3).\end{aligned} \tag{5.36}$$

6. Convariance functions. Let us now compute the lateral convariance functions for the logarithm of the relative amplitude

$$C_{LL}(\tau) = \left\langle\left[\log\frac{A(\mathbf{r})}{A_0(\mathbf{r})} - \left\langle\log\frac{A(\mathbf{r})}{A_0(\mathbf{r})}\right\rangle\right]\left[\log\frac{A(\mathbf{r}+\boldsymbol{\tau})}{A_0(\mathbf{r}+\boldsymbol{\tau})} - \left\langle\log\frac{A(\mathbf{r}+\boldsymbol{\tau})}{A_0(\mathbf{r}+\boldsymbol{\tau})}\right\rangle\right]\right\rangle \tag{6.1}$$

and of the phase fluctuation

$$C_{\phi\phi}(\boldsymbol{\tau}) = \langle[\phi(\mathbf{r}) - \langle\phi(\mathbf{r})\rangle]\cdot[\phi(\mathbf{r}+\boldsymbol{\tau}) - \langle\phi(\mathbf{r}+\boldsymbol{\tau})\rangle]\rangle \tag{6.2}$$

where $\boldsymbol{\tau}$ is a radial position vector in the plane normal to the z-axis. These can be expressed as

$$C_{LL}(\boldsymbol{\tau}) = \tfrac{1}{2}\,\mathrm{Re}\,[C_{\psi\psi^*}(\boldsymbol{\tau}) + C_{\psi\psi}(\boldsymbol{\tau})], \tag{6.3}$$

$$C_{\phi\phi}(\boldsymbol{\tau}) = \tfrac{1}{2}\,\mathrm{Re}\,[C_{\psi\psi^*}(\boldsymbol{\tau}) - C_{\psi\psi}(\boldsymbol{\tau})] \tag{6.4}$$

where

$$C_{\psi\psi^*}(\boldsymbol{\tau}) = \langle[\psi(\mathbf{r}) - \langle\psi(\mathbf{r})\rangle]\cdot[\psi^*(\mathbf{r}+\boldsymbol{\tau}) - \langle\psi^*(\mathbf{r}+\boldsymbol{\tau})\rangle]\rangle, \tag{6.5}$$

$$C_{\psi\psi}(\boldsymbol{\tau}) = \langle[\psi(\mathbf{r}) - \langle\psi(\mathbf{r})\rangle]\cdot[\psi(\mathbf{r}+\boldsymbol{\tau}) - \langle\psi(\mathbf{r}+\boldsymbol{\tau})\rangle]\rangle \tag{6.6}$$

which shall be computed to the second order in ϵ from our explicit solution (3.25). Now from (3.20), it follows that $\langle\psi_1(r)\rangle = 0$, since ψ_1 is linear in μ and since we assume that $\langle\mu\rangle = 0$. Hence

$$C_{\psi\psi^*}(\boldsymbol{\tau}) = \epsilon^2\langle\psi_1(\mathbf{r})\psi_1^*(\mathbf{r}+\boldsymbol{\tau})\rangle + O(\epsilon^3), \tag{6.7}$$

$$C_{\psi\psi}(\boldsymbol{\tau}) = \epsilon^2\langle\psi_1(\mathbf{r})\psi_1(\mathbf{r}+\boldsymbol{\tau})\rangle + O(\epsilon^3). \tag{6.8}$$

The covariance of $\psi_1(\mathbf{r})$ shall be computed from (3.20) and the approximations implicit in (5.12) and (5.13). Thus we represent $\psi_1(\mathbf{r})$ as in (5.14):

$$\psi_1(\mathbf{r}) = -\frac{ik}{\pi}\int_0^z \frac{dz_0}{\gamma(z, z_0)}\int d\boldsymbol{\varrho}_0\, \mu(\mathbf{r}_0)\exp\left[\frac{(\boldsymbol{\varrho}_0 - \boldsymbol{\varrho}Z_0/Z)^2}{\gamma(z, z_0)}\right]. \tag{6.9}$$

Hence

$$\operatorname{Re} C_{\psi\psi^*}(\boldsymbol{\tau}) = \frac{\epsilon^2k^2}{\pi^2}\operatorname{Re}\int_0^z\int_0^z \frac{dz_1\, dz_2}{\gamma(z, z_1)\gamma^*(z, z_2)}\int d\boldsymbol{\varrho}_1\int d\boldsymbol{\varrho}_2\, \langle\mu(\mathbf{r}_1)\mu(\mathbf{r}_2)\rangle$$
$$\cdot\exp\left[\frac{(\boldsymbol{\varrho}_1 - \boldsymbol{\varrho}Z_1/Z)^2}{\gamma(z, z_1)} + \frac{(\boldsymbol{\varrho}_2 - (\boldsymbol{\varrho} + \boldsymbol{\tau})Z_2^*/Z^*)^2}{\gamma^*(z, z_2)}\right] + O(\epsilon^3), \tag{6.10}$$

$$\operatorname{Re} C_{\psi\psi}(\boldsymbol{\tau}) = -\frac{\epsilon^2k^2}{\pi^2}\operatorname{Re}\int_0^z\int_0^z \frac{dz_1\, dz_2}{\gamma(z, z_1)\cdot\gamma(z, z_2)}\int d\boldsymbol{\varrho}_1\int d\boldsymbol{\varrho}_2\, \langle\mu(\mathbf{r}_1)\mu(\mathbf{r}_2)\rangle$$
$$\cdot\exp\left[\frac{(\boldsymbol{\varrho}_1 - \boldsymbol{\varrho}Z_1/Z)^2}{\gamma(z, z_1)} + \frac{(\boldsymbol{\varrho}_2 - (\boldsymbol{\varrho} + \boldsymbol{\tau})Z_2/Z)^2}{\gamma(z, z_2)}\right] + O(\epsilon^3). \tag{6.11}$$

Upon inserting (4.14) into (6.10) and (6.11), and introducing the change of variable (5.18), we find

$$\operatorname{Re} C_{\psi\psi^*}(\boldsymbol{\tau}) = \frac{k^2}{\pi^2}\operatorname{Re}\int_0^z\int_0^z \frac{dz_1\, dz_2}{\gamma(z, z_1)\cdot\gamma^*(z, z_2)}\int d\boldsymbol{\zeta}\, C_n[|(z_1 - z_2)^2 + \zeta^2|^{1/2}; s]$$
$$\cdot\int d\mathbf{n}\exp\left[\frac{(\mathbf{n} + \boldsymbol{\zeta}/2 - \boldsymbol{\varrho}Z_1/Z)^2}{\gamma(z, z_1)} + \frac{(\mathbf{n} - \boldsymbol{\zeta}/2 - (\boldsymbol{\varrho} + \boldsymbol{\tau})Z_2^*/Z^*)^2}{\gamma^*(z, z_2)}\right] + O(\epsilon^3), \tag{6.12}$$

$$\operatorname{Re} C_{\psi\psi}(\boldsymbol{\tau}) = -\frac{k^2}{\pi^2}\operatorname{Re}\int_0^z\int_0^z \frac{dz_1\, dz_2}{\gamma(z, z_1)\cdot\gamma(z, z_2)}\int d\boldsymbol{\zeta}\, C_n[|(z_1 - z_2)^2 + \zeta^2|^{1/2}; s]$$
$$\cdot\int d\mathbf{n}\exp\left[\frac{(\mathbf{n} + \boldsymbol{\zeta}/2 - \boldsymbol{\varrho}Z_1/Z)^2}{\gamma(z, z_1)} + \frac{(\mathbf{n} - \boldsymbol{\zeta}/2 - (\boldsymbol{\varrho} + \boldsymbol{\tau})Z_2/Z)^2}{\gamma(z, z_2)}\right] + O(\epsilon^3). \tag{6.13}$$

Using the result given by (5.21) permits the $d\mathbf{n}$ integration to be performed explicitly

$$\operatorname{Re} C_{\psi\psi^*}(\boldsymbol{\tau}) = -\frac{k^2}{\pi}\operatorname{Re}\int_0^z\int_0^z \frac{dz_1\, dz_2}{\gamma(z, z_1) + \gamma^*(z, z_2)}\int d\boldsymbol{\zeta}\, C_n[|(z_1 - z_2)^2 + \zeta^2|^{1/2}; s]$$
$$\cdot\exp\left[\frac{(\boldsymbol{\zeta} - \boldsymbol{\varrho}Z_1/Z + (\boldsymbol{\varrho} + \boldsymbol{\tau})Z_2^*/Z^*)^2}{\gamma(z, z_1) + \gamma^*(z, z_2)}\right] + O(\epsilon^3), \tag{6.14}$$

$$\operatorname{Re} C_{\psi\psi}(\boldsymbol{\tau}) = \frac{k^2}{\pi}\operatorname{Re}\int_0^z\int_0^z \frac{dz_1\, dz_2}{\gamma(z, z_1) + \gamma(z, z_2)}\int d\boldsymbol{\zeta}\, C_n[|(z_1 - z_2)^2 + \zeta^2|^{1/2}; s]$$
$$\cdot\exp\left[\frac{(\boldsymbol{\zeta} - \boldsymbol{\varrho}Z_1/Z + (\boldsymbol{\varrho} + \boldsymbol{\tau})Z_2/Z)^2}{\gamma(z, z_1) + \gamma(z, z_2)}\right] + O(\epsilon^3). \tag{6.15}$$

Assuming that the observation point $\mathbf{r} = \mathbf{z}$ is in the far field, i.e., $z \gg |k\alpha^2|$, then examination of the terms in the exponentials of (6.14) and (6.15) shows that the dependence on $\boldsymbol{\varrho}$ is quite weak when $\rho \ll z$. We shall therefore set $\boldsymbol{\varrho} = 0$. Introducing the change of variables (5.27) and extending the limits of the resulting dt integration between $-\infty$ and $+\infty$ yields

$$\operatorname{Re} C_{\psi\psi^*}(\boldsymbol{\tau}) = -\frac{k^2}{\pi}\operatorname{Re}\int_0^z ds\int_{-\infty}^{+\infty} dt\int d\boldsymbol{\zeta}\, \frac{C_n[|t^2 + \zeta^2|^{1/2}; s]}{\gamma(z, s + t/2) + \gamma^*(z, s - t/2)}$$
$$\cdot\exp\left[\frac{(\boldsymbol{\zeta} + \boldsymbol{\tau}(s - t/2 + ik\alpha^{*2})/Z^*)^2}{\gamma(z, s + T/2) + \gamma^*(z, s - t/2)}\right] + O(\epsilon^3), \tag{6.16}$$

$$\operatorname{Re} C_{\psi\psi}(\tau) = \frac{k^2}{\pi} \operatorname{Re} \int_0^z ds \int_{-\infty}^{+\infty} dt \int d\zeta \frac{C_n[|t^2 + \zeta^2|^{1/2}; s]}{\gamma(z, s + t/2) + \gamma(z, s - t/2)} \cdot \exp\left[\frac{(\zeta + \tau(s - t/2 - ik\alpha^2)/Z)^2}{\gamma(z, s + t/2) + \gamma(z, s - t/2)}\right] + O(\epsilon^3). \tag{6.17}$$

Now

$$\gamma(z, s + t/2) + \gamma^*(z, s - t/2) = \frac{2[(z - s)^2 + t^2/4] \operatorname{Re}(k\alpha^2) + it[z^2 - k^2 |\alpha|^4 - 2s(z + \operatorname{Im}(k\alpha^2))]}{-kZZ^*/2} \tag{6.18}$$

and

$$\gamma(z, s + t/2) + \gamma(z, s - t/2) = 2\gamma(z, s) - t^2/ikZ. \tag{6.19}$$

Since $C_n[(t^2 + \zeta^2)^{1/2}, s]$ falls to zero rapidly for values of t beyond a correlation distance a_c, the dt integrations in (6.16) and (6.17) are significant in the rather limited range $0 < t < a_c$. Hence the coefficients (6.18) and (6.19) have a very weak dependence on t. Hence there is negligible error introduced by setting $t = 0$ in (6.18), (6.19), and in the exponential terms of (6.16) and (6.17). If we do this, and introduce the function

$$f_c(\zeta, s) = \int_0^\infty C_n[|t^2 + \zeta^2|^{1/2}; s]\, dt \tag{6.20}$$

then

$$\operatorname{Re} C_{\psi\psi^*}(\tau) = -\frac{k^2}{\pi} \operatorname{Re} \int_0^z ds \int d\zeta \frac{f_c(\zeta, s)}{\operatorname{Re}\gamma(z, s)} \cdot \exp\left[\frac{(\zeta + \tau(s - ik\alpha^2)^*/Z^*)^2}{2 \operatorname{Re}\gamma(z, s)}\right] + O(\epsilon^3), \tag{6.21}$$

$$\operatorname{Re} C_{\psi\psi}(\tau) = \frac{k^2}{\pi} \operatorname{Re} \int_0^z ds \int d\zeta \frac{f_c(\zeta, s)}{\gamma(z, s)} \exp\left[\frac{(\zeta + \tau(s - ik\alpha^2)/Z)^2}{2\gamma(z, s)}\right] + O(\epsilon^3). \tag{6.22}$$

Therefore, from (6.3) and (6.4), with $S = s - ik\alpha^2$,

$$C_{LL}(\tau) = \frac{k^2}{2\pi} \operatorname{Re} \int_0^z ds \int d\zeta\, f_c(\zeta, s) \cdot\left[\frac{1}{\gamma(z, s)} \exp\left[\frac{(\zeta + \tau S/Z)^2}{2\gamma(z, s)}\right] - \frac{1}{\operatorname{Re}\gamma(z, s)} \exp\left[\frac{(\zeta + \tau S^*/Z^*)^2}{2 \operatorname{Re}\gamma(z, s)}\right]\right] + O(\epsilon^3), \tag{6.23}$$

$$C_{\phi\phi}(\tau) = -\frac{k^2}{2\pi} \operatorname{Re} \int_0^z ds \int d\zeta\, f_c(\zeta, s) \cdot\left[\frac{1}{\gamma(z, s)} \exp\left[\frac{(\zeta + \tau S/Z)^2}{2\gamma(z, s)}\right] + \frac{1}{\operatorname{Re}\gamma(z, s)} \exp\left[\frac{(\zeta + \tau S^*/Z^*)^2}{2 \operatorname{Re}\gamma(z, s)}\right]\right] + O(\epsilon^3). \tag{6.24}$$

To obtain the corresponding structure functions, we introduce the function

$$\begin{aligned} f_D(\zeta, s) &= \int_0^\infty D_n[(t^2 + \zeta^2)^{1/2}; s] - D_n(t, s)\, dt \\ &= 2[f_c(0, s) - f_c(\zeta, s)] \end{aligned} \tag{6.25}$$

whence from (4.10), (6.23), and (6.24)

$$D_{LL}(\boldsymbol{\tau}) = \frac{k^2}{\pi} \operatorname{Re} \int_0^z ds \int d\boldsymbol{\zeta}\, f_D(\zeta, s)$$
$$\cdot\left\{\frac{1}{\gamma(z, s)}\left(\exp\left[\frac{\zeta^2}{2\gamma(z, s)}\right] - \exp\left[\frac{(\boldsymbol{\zeta} + \boldsymbol{\tau}S/Z)^2}{2\gamma(z, s)}\right]\right)\right.$$
$$\left. - \frac{1}{\operatorname{Re}\gamma(z, s)}\left(\exp\left[\frac{\zeta^2}{2\operatorname{Re}\gamma(z, s)}\right] - \exp\left[\frac{(\boldsymbol{\zeta} + \boldsymbol{\tau}S^*/Z^*)^2}{2\operatorname{Re}\gamma(z, s)}\right]\right)\right\} + O(\epsilon^3), \tag{6.26}$$

$$D_{\phi\phi}(\boldsymbol{\tau}) = -\frac{k^2}{\pi} \operatorname{Re} \int_0^z ds \int d\boldsymbol{\zeta}\, f_D(\zeta, s)$$
$$\cdot\left\{\frac{1}{\gamma(z, s)}\left(\exp\left[\frac{\zeta^2}{2\gamma(z, s)}\right] - \exp\left[\frac{(\boldsymbol{\zeta} + \boldsymbol{\tau}S^*/Z^*)^2}{2\gamma(z, s)}\right]\right)\right.$$
$$\left. + \frac{1}{\operatorname{Re}\gamma(z, s)}\left(\exp\left[\frac{\zeta^2}{2\operatorname{Re}\gamma(z, s)}\right] - \exp\left[\frac{(\boldsymbol{\zeta} + \boldsymbol{\tau}S^*/Z^*)^2}{2\operatorname{Re}\gamma(z, s)}\right]\right)\right\} + O(\epsilon^3). \tag{6.27}$$

The variance of the logarithm of the relative amplitude and phase is obtained from (6.23) and (6.24) with $\boldsymbol{\tau} = 0$.

7. Spectral representations. The means, variances, and convariances computed above may be cast into an alternative form of representation which depends upon the spectral function Φ defined by a three dimensional Fourier integral of C_n :

$$\Phi(\sigma; s) = \int d\mathbf{r} \exp(i\boldsymbol{\sigma}\cdot\mathbf{r})C_n(r; s) \tag{7.1}$$

rather than on the "f functions" (5.33), (6.20), and (6.25). By using Fourier's repeated integral identity, C_n may be expressed as

$$C_n(r; s) = \frac{1}{(2\pi)^3}\int d\boldsymbol{\sigma} \exp(-i\boldsymbol{\sigma}\cdot\mathbf{r})\Phi(\sigma; s). \tag{7.2}$$

Since

$$\nabla^2 C_n(r; s) = \frac{-1}{(2\pi)^3}\int d\boldsymbol{\sigma} \exp(-i\boldsymbol{\sigma}\cdot\mathbf{r})\sigma^2\Phi(\sigma; s) \tag{7.3}$$

then

$$\begin{aligned} f_0(\zeta; s) &= \frac{-1}{2}\int_{-\infty}^{+\infty} dz \frac{1}{(2\pi)^3}\int d\boldsymbol{\sigma} \exp(-i\boldsymbol{\sigma}\cdot\mathbf{r})\sigma^2\Phi(\sigma; s), \\ &= \frac{-1}{8\pi^2}\int d\boldsymbol{\sigma} \exp(-i\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\sigma^2\Phi(\sigma; s). \end{aligned} \tag{7.4}$$

Similarly

$$f_c(\zeta; s) = \frac{1}{8\pi^2}\int d\boldsymbol{\sigma} \exp(-i\boldsymbol{\sigma}\cdot\boldsymbol{\zeta})\Phi(\sigma; s). \tag{7.5}$$

Substituting (7.4) into (5.36), interchanging the order of the integrations and performing the resulting $d\zeta$ integration gives

$$\langle\psi(z)\rangle = \frac{ik}{2}\int_0^z C_n(0; s)\, ds - \frac{ik}{8\pi}\int_0^s ds \int_0^s F[-\gamma(s, q)/2; q]\, dq + O(\epsilon^3) \tag{7.6}$$

where we have introduced F, the Laplace transform of $\sigma\Phi(\sigma^{1/2}; q)$:

$$F(p; q) = \mathfrak{L}[\sigma\Phi(\sigma^{1/2}; q)] \equiv \int_0^\infty \sigma\Phi(\sigma^{1/2}; q) \exp(-p\sigma)\, d\sigma. \tag{7.7}$$

Likewise, we insert (7.5) into (6.23) and (6.24) and perform the $d\zeta$ integration, whereupon rearranging, we find

$$C_{LL}(\tau) = -\frac{k^2}{8\pi} \operatorname{Re} \int_0^z ds \int_0^\infty \Phi(\sigma^{1/2}; s) J_0(\sigma^{1/2}\tau S/Z) \exp[\sigma\gamma(z, s)/4] \cdot [\exp[\sigma\gamma(z, s)/4] - \exp[\sigma\gamma^*(z, s)/4]]\, d\sigma + O(\epsilon^3), \tag{7.8}$$

$$C_{\phi\phi}(\tau) = \frac{k^2}{8\pi} \operatorname{Re} \int_0^z ds \int_0^\infty \Phi(\sigma^{1/2}; s) J_0(\sigma^{1/2}\tau S/Z) \exp[\sigma\gamma(z, s)/4] \cdot [\exp[\sigma\gamma(z, s)/4] + \exp[\sigma\gamma^*(z, s)/4]]\, d\sigma + O(\epsilon^3). \tag{7.9}$$

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References

1. V. I. Tatarski, *Wave propagation in a turbulent medium,* transl by R. A. Silverman, McGraw-Hill, New York, 1960
2. D. L. Fried, *The statistics of a geometric interpretation of wavefront distortion,* J. Opt. Soc. Am., Vol. 55, No. 11, pp. 1427–1435, 1965.
3. D. L. Fried, *The effect of wavefront distortion on the performance of an ideal optical heterodyne receiver and an ideal camera,* Conference on Atmospheric Limitations to Optical Propagation, Central Radio Propagation Lab., National Bureau of Standards, and National Center for Atmospheric Research, Boulder, Colorado, March 1965
4. L. A. Chernov, *Wave propagation in a random medium,* English transl, McGraw-Hill, New York, 1960
5. W. C. Hoffman, *Wave propagation in a general random continuous medium,* Proc. Sympos. Appl. Math. Vol. 16, pp. 117–144, Amer. Math. Soc., Providence, R. I., 1964