PSEUDO-SIMILARITY SOLUTIONS OF THE ONE-DIMENSIONAL DIFFUSION EQUATION WITH APPLICATIONS TO THE PHASE CHANGE PROBLEM*

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Abstract. New solutions of the diffusion equation may be used to prescribe both the diffusion potential and the diffusional flow rate, along the moving curve $X = \alpha(1 + \beta \cdot T)^{1/2}$, as arbitrary power series in the variable $(\alpha(1 + \beta \cdot T)^{1/2})$, where α and β are arbitrary constants and T is time.

The new solutions are applicable to problems with planar, cylindrical, or spherical symmetry. They include, as special cases, the instantaneous source solutions of the diffusion equation, the classical closed-form solutions to the phase change problem, and new solutions to the phase-change problem for nonuniform initial conditions.

A. Introduction. We treat the one-dimensional diffusion of heat perpendicular to the surfaces of parallel planes, coaxial cylinders, and concentric spheres. The heat flow between bounding surfaces of these types is governed by the following partial differential equation:

$$X^{-k} \cdot (X^k \cdot K \cdot U_X)_X = R \cdot C \cdot U_T(X, T). \tag{1}$$

Here $U(X, T)$ is the temperature at point X at time T, $K(U)$ is the thermal conductivity, $C(U)$ is the specific heat, R is the constant density of the medium, and D is the constant diffusivity of the medium, $D \equiv K/RC$. Also, $k = 0, 1, 2$ for plane, cylinder, and sphere, respectively. If $k = 1$, X is measured from the axis of a cylinder. If $k = 2$, X is measured from the center of a sphere.

We assume that Eq. (1) has a solution which is of a particular form; we obtain a solution which is useful for treating phase change problems.

1. The phase change problem. Phase change is assumed to occur at the constant temperature U^*. The location $X = Y(T)$ of a moving phase boundary is then fixed by the implicit relation

$$U(X, T) = U^* = \text{constant} \quad \text{at} \quad X = Y(T). \tag{2a}$$

Latent heat of phase change is absorbed or released at $X = Y(T)$, depending on whether heating or cooling is occurring. In either case:

> the velocity dY/dT of the phase boundary is proportional to the net rate at which heat enters the phase boundary from the two sides of that boundary. (2b)

* Received March 12, 1965; revised manuscript received June 13, 1966. This is based on a dissertation [9] accepted by Rensselaer Polytechnic Institute in partial fulfillment of the requirements for the degree of Doctor of Engineering Science. The author wishes to acknowledge valuable discussions with Professor Bronis R. Onuf (who suggested the thesis topic), Professor Howard W. Butler (who acted as thesis adviser) and Professor Murray R. Spiegel (who served on the Doctoral Committee). The research was supported by an educational grant from Pratt & Whitney Aircraft Division of United Aircraft Corporation.

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Any solution $U(X, T)$ to a phase change problem must satisfy some prescribed condition at $T = 0$. The solution $U(X, T)$ may also be required to satisfy a condition at some fixed boundary, say at $X = 0$ or at $X = Y(0)$—this is the "direct" problem in which $Y(T)$ is to be determined as part of the solution. As an alternate, $Y(T)$ may be prescribed, this being the "inverse" problem. We solve the inverse problem for $Y(T) \cdot Y_T(T) = \text{constant}$, with $k = 0, 1, 2$.

2. Connection with prior work. For $Y(T) \cdot Y_T(T) = \text{constant}$, the first closed form solution to $U(X, T)$ was published by Lamé and Clapeyron [6] in 1831. These authors dealt only with the planar case $k = 0$. For $k = 1$ and 2, closed form solutions were derived by Rieck [5] in 1924 and re-derived by Frank [4] in 1950. These early solutions have wide application [2] but are restricted to the case $U(X, 0) = \text{constant}$.

For $Y(T) \cdot Y_T(T) = \text{constant}$, with $U(X, 0) \neq \text{constant}$, solutions to $U(X, T)$ were given by Redozubov [11] in 1962 for $k = 0$ and by Martynov [10] in 1960 for $k = 2$. For the same case we use a different method to develop solutions for $k = 0, 1, 2$.*

The early solutions were functions only of the similarity parameter $X/Y(T)$. Our solutions are functions of $X/Y(T)$ and T.

B. The form assumed for the solutions. We begin by writing the one-dimensional diffusion Eq. 1 in the dimensionless form

$$x^{-k} \cdot (x^k \cdot u_x)_x = u_t(x, t), \tag{3a}$$

where

$$u(x, t) \equiv (1/L) \cdot \int_{U'=U^*}^{U'=U(X,T)} C(U') \cdot dU'. \tag{3b}$$

Here $|L|$ = energy per unit mass necessary to cause phase change, with $L > 0$ for heating and $L < 0$ for cooling. Also,

$$x \equiv X/X^* \quad \text{and} \quad t \equiv TD/(X^*)^2, \tag{3c}$$

where X^* is an arbitrary reference length. We now assume that Eq. (3a) has a solution of the form

$$u(x, t) = \phi(z) \cdot \psi(t), \qquad z \equiv b^2 x^2 / y^2(t), \qquad b = \text{constant}, \tag{3d}$$

with $\phi(z)$, $\psi(t)$, and $y(t)$ being unknown, and with b being arbitrary. We substitute this into Eq. (3a) and obtain

$$a(z, t) \equiv (y^2/4b^2) \cdot \frac{\psi_t(t)}{\psi(t)} = \frac{z \cdot \phi_{zz}}{\phi(z)} + \left[\frac{1+k}{2} + (yy_t/2b^2) \cdot z\right] \cdot \frac{\phi_z}{\phi(z)}. \tag{3e}$$

The variables z and t are clearly separable if $yy_t = \text{constant}$. We therefore assume that

$$yy_t = \text{constant} = y(0) \cdot y_t(0), \tag{4a}$$

whence

$$y(t) = y_0 \cdot (1 + 2y_1 t/y_0)^{1/2}, \tag{4b}$$

*The condition $Y \cdot Y_T = \text{constant}$ is one special case of the more general condition $Y_{TT}/Y_T^3 = \text{constant} = \gamma$. For $\gamma = 0$, closed form solutions to $U(X, T)$ were given by Stefan [12] in 1890 for $k = 0$ and by Langford [7, 9] in 1965–1966 for $k = 2$. For $\gamma \neq 0$, a closed form solution was given by Langford [8, 9] in 1965–1966 for $k = 2$. These solutions are restricted to the case $U(X, 0) = U^*$.

and

$$y_t(t) = y_1/(1 + 2y_1t/y_0)^{1/2}, \tag{4c}$$

where y_0 and y_1 are arbitrary constants.

We now have $a(z, t) = a =$ constant. We further simplify Eq. (3e) by choosing $b^2 = -\frac{1}{2}y_0y_1$ so that $yy_t/2b^2 = -1$. We then integrate and set $\psi(0) = 1$ to obtain

$$u(x, t) = \phi(z)\cdot(y^2/y_0^2)^{-a}, \tag{5a}$$

where

$$z = b^2x^2/y^2, \qquad b^2 = -\tfrac{1}{2}y_0y_1\ , \tag{5b}$$

and

$$y(t) = y_0\cdot(1 + 2y_1t/y_0)^{1/2}, \tag{5c}$$

with y_0 and y_1 being arbitrary constants, and with $\phi(z)$ being a solution of

$$z\cdot\phi_{zz} + (c - z)\cdot\phi_z = a\cdot\phi(z), \qquad c \equiv (1 + k)/2, \qquad k = 0, 1, 2. \tag{5d}$$

1. The confluent hypergeometric equation. Equation (5d) is one of the standard forms of the confluent hypergeometric equation [1]. One of its solutions is the Kummer function $M(a, c, z)$ which may be defined thus for arbitrary a, c, z:

$$M(a, c, z) \equiv 1 + \frac{a}{c}\cdot\frac{z}{1!} + \frac{a(a+1)}{c(c+1)}\cdot\frac{z^2}{2!} + \frac{a(a+1)(a+2)}{c(c+1)(c+2)}\cdot\frac{z^3}{3!} + \cdots . \tag{6a}$$

Now define

$$W(a, c, z) \equiv (z^{1-c})\cdot M(a + 1 - c, 2 - c, z), \qquad c \neq 1, 2, 3, \cdots . \tag{6b}$$

This function, if it exists and differs from $M(a, c, z)$, is a solution of Eq. (5d) which is linearly independent of $M(a, c, z)$. For $c = 1, 2, 3, \cdots$, a solution of Eq. (5d) which is linearly independent of $M(a, c, z)$ will in general involve a term in $\ln_e(z)$. It will be denoted by $W(a, c, z)$ [1].

The complete solution to Eq. (5d) may now be given as

$$\phi(z) = m\cdot M(a, c, z) + w\cdot W(a, c, z), \qquad m \text{ and } w \text{ constants}. \tag{7a}$$

For later use we note that, for $z = 0$, corresponding to $x = 0$, it can be shown that:

$$\lim_{x\to0}\,[M(a, c, z)] = 1, \qquad \lim_{x\to0}\,[dM(a, c, z)/dx] = 0; \tag{7b}$$

$$\begin{aligned} \lim_{x\to0}\,[W(a, c, z)] &= 0 \quad \text{for} \quad c = 1/2 \quad (k = 0),\\ &= \infty \quad \text{for} \quad c = 1, 3/2 \quad (k = 1, 2); \end{aligned} \tag{7c}$$

$$\begin{aligned} \lim_{x\to0}\,[x^k\cdot W_x(a, c, z)] &= b/y(t) \quad \text{for} \quad c = 1/2 \quad (k = 0),\\ &= \text{constant} \quad \text{for} \quad c = 1 \quad (k = 1),\\ &= -y(t)/b \quad \text{for} \quad c = 3/2 \quad (k = 2). \end{aligned} \tag{7d}$$

We will now examine the properties of $\phi(z)$ in order that we may find a more useful form for the solution (5a).

2. Orthogonality relations. Equation (5d) may be rewritten in the form

$$(z^c\cdot e^{-z}\cdot\phi_z)_z = a\cdot z^{c-1}\cdot e^{-z}\cdot\phi(z). \tag{8a}$$

This equation is of such a nature that we can require that its solution $\phi(z)$ satisfy the boundary conditions

$$j_1 \cdot \phi(z_1) + J_1 \cdot \phi_z(z_1) = 0, \tag{8b}$$

and

$$j_2 \cdot \phi(z_2) + J_2 \cdot \phi_z(z_2) = 0, \tag{8c}$$

where j_i, J_i, and z_i are real constants, chosen independent of the separation constant a.

Equations 8 form a Sturm-Liouville system [3]. Prescribed values of the j_i, J_i, z_i lead to an infinite number of values of the parameter a for which the system 8 has a nontrivial solution. These values of a are characteristic of the system. Further, all the characteristic values are real and not more than a finite number of them are positive. For each characteristic value a there is a characteristic function $\phi^a(z)$ which is a solution of system 8. Now suppose that $\phi'(z)$ and $\phi''(z)$ are solutions of the Sturm–Liouville system 8 corresponding to the characteristic values $a = s'$ and $a = s''$, respectively. Then from equation 8a it follows that

$$(z^c \cdot e^{-z} \cdot \phi'_z)_z = s' \cdot z^{c-1} \cdot e^{-z} \cdot \phi'(z) \tag{9a}$$

and

$$(z^c \cdot e^{-z} \cdot \phi''_z)_z = s'' \cdot z^{c-1} \cdot e^{-z} \cdot \phi''(z). \tag{9b}$$

These equations can be used with the boundary conditions (8b) and (8c) to show that

$$(s' - s'') \int_{z_1}^{z_2} (z^{c-1} \cdot e^{-z}) \cdot \phi'(z) \cdot \phi''(z) \cdot dz = 0. \tag{10}$$

Let the set of characteristic values be denoted by s_n; $a = s_0, s_1, s_2, \cdots$. Equation 10 shows that the corresponding set $\phi^{s_n}(z)$ of characteristic functions is orthogonal with respect to the weighting function $z^{c-1} \cdot e^{-z}$ over the interval $0 < z_1 \leqq z \leqq z_2$. This set of orthogonal functions can easily be made to yield the generalized Fourier series expansion of any function with sectionally continuous first derivatives [3]. The series expansion can be used with Eqs. (5a) and (7a) to show that the final solution $u(x, t)$ of the diffusion Eq. (3a) will have the form

$$u(x, t) = \sum_{a=s_0}^{a=s_\infty} \{[m_a \cdot M(a, c, z) + w_a \cdot W(a, c, z)] \cdot (y^2/y_0^2)^{-a}\}, \tag{11a}$$

where m_a and w_a are constants, i.e., they depend only on a. Here (11b)

$$z = b^2x^2/y^2, \qquad b^2 = -\tfrac{1}{2}y_0 y_1, \qquad c = (1 + k)/2, \tag{11c}$$

and

$$y(t) = y_0 \cdot (1 + 2y_1 t/y_0)^{1/2}, \tag{11d}$$

with y_0 and y_1 arbitrary constants.

The functions (11a) is formally a solution of Eq. (3a) for any sequences of constants for the values of the s_n, m_{s_n}, and w_{s_n}. It is easily shown that these sequences of constants can be chosen such that both $u(y(t), t)$ and $u_x(y(t), t)$ are arbitrary power series in the variable $y(t)$. This makes it possible to use the solution (11a) to solve the special class of phase change problems for which the motion of the phase boundary is given by Eq. (11d).* (See Footnote Next Page)

C. The phase change problem. The solution (11a) will be used to develop the classical solutions to the phase change problem; it will then be used to develop the solution to the ablation problem for which $u(x, 0) \neq$ constant. The solution to the ablation problem will be used to develop the solution to the melting problem for which $u(x, 0) \neq$ constant. The results are applicable for $k = 0, 1, 2$ *provided* the phase boundary $x = y(t)$ moves such that $yy_t =$ constant.

1. The classical solutions. The classical closed from solutions to the phase change problem may be obtained from equation (11a) by assuming that the only nonzero coefficient is w_{s_0} and that $s_0 = 0$. We make the indicated assumptions, add a constant term, and introduce a new multiplicative constant v^*. We obtain

$$u(x, t) = 2\cdot(v^*)\cdot b^{k+1}\cdot \exp[-b^2]\cdot\int_{Z=b}^{Z=bx/y} Z^{-k}\cdot\exp[+Z^2]\cdot dZ. \tag{12a}$$

This may also be written in the form

$$u(x, t) = 2\cdot(v^*)\cdot B^{k+1}\cdot \exp[+B^2]\cdot\int_{Z=Bx/y}^{z=B} Z^{-k}\cdot\exp[-Z^2]\cdot dZ, \tag{12b}$$

where

$$B^2 \equiv -b^2 = \tfrac{1}{2}y_0y_1, \quad \text{and} \quad y = (y_0^2 + 2y_0y_1t)^{1/2} = (y_0^2 + 4B^2t)^{1/2}. \tag{12c}$$

For the solution (12a) or (12b),

$$u(y(t), t) = 0, \qquad u_x(y(t), t) = -(v^*)\cdot y_t(t). \tag{12d}$$

From this it follows that the curve $x = y(t)$ gives the location of the moving isotherm $u = 0$; and that the velocity $y_t(t)$ of the isotherm is proportional to the temperature gradient at the isotherm.

Equations (12a) and (12b) can be called the "classical solutions" to the phase change problem because they include most of the closed form solutions used in studies of the phase change problem [2]. To obtain the classical solutions to the phase change problem, we must set $y_0 = 0$ in Eq. (12c). We obtain $y(t) = 2Bt^{1/2}$, whence $y_t(t) = B/t^{1/2}$, and $y_t(0) = \infty$. Corresponding to the infinite velocity at zero time, the temperature $f(t)$ at the point $x = y_0 = 0$ undergoes a step change from a value zero at zero time to some constant value

$$f(t) = 2\cdot(v^*)\cdot B^{k+1}\cdot\exp[B^2]\cdot\int_0^B Z^{-k}\cdot\exp[-z^2]\cdot dZ = \text{constant}, \quad t > 0; \tag{13a}$$

$$f(t) = \infty \quad \text{for} \quad k = 1 \text{ or } 2 \text{ and } t > 0. \tag{13b}$$

2. The ablation problem with inward motion. Suppose by way of example that a slab, cylinder, or sphere is being symmetrically heated by some surface heat source and that at zero time the temperature at the surface $x = y_0$ is equal to the ablation temperature $u = 0$. Then

$$u(y_0, 0) = 0. \tag{14a}$$

We take

$$u(x, 0) = I(x) = \text{a prescribed function}, \qquad 0 \leqq x \leqq y_0, \qquad I(x) \leqq 0, \tag{14b}$$

*The solution (11a) can also be used to generate the instantaneous plane, line, and point source solutions to the diffusion equation. The procedure is simply to assume that the only nonzero coefficient in equation (11a) is m_{s_0}, set $m_{s_0} = 1$, set $s_0 = c$, and replace t by $(t - y_0/2y_1)$.

with $I(x)$ assumed to have sectionally continuous first derivatives. We now assume there is no heat source acting at $x = 0$ so, by symmetry, we have

$$u_x(0, t) = 0. \tag{14c}$$

We assume further that the location $x = y(t)$ of the moving ablation surface $u = 0$ is given by Eq. (11d). Then

$$u(y(t), t) = 0, \qquad y(t) = y_0 \cdot (1 + 2y_1 t/y_0)^{1/2}, \qquad y_1^2 \text{ arbitrary}. \tag{14d}$$

Our problem is to find a solution $u(x, t)$ of the diffusion Eq. (3a) such that $u(x, t)$ satisfies the initial condition (14b) and the boundary conditions (14c) and (14d), with $0 \leqq x \leqq y(t)$, $t > 0$.

We assume that $u(x, t)$ is given by Eq. (11). Condition (14c) can be used with Eqs. (11a), (7b), and (7d) to show that all the w_{s_n} in Eq. (11a) must have the value zero. Equation (11a) can therefore be written

$$u(x, t) = \sum_{a=s_0}^{a=s_\infty} m_a \cdot M(a, c, z) \cdot (y^2/y_0^2)^{-a}, \qquad z = b^2x^2/y^2. \tag{15a}$$

Use of this result with condition (14d) gives rise to the following eigenvalue condition for the determination of the s_n :

$$M(a, c, b^2) = 0, \qquad a = s_0, s_1, s_2, \cdots; \qquad b^2 \equiv -\tfrac{1}{2}y_0 y_1 . \tag{15b}$$

With the s_n known, we may use Eq. (15a) with the initial condition (14b) to show that

$$u(x, 0) = \sum_{a=s_0}^{a=s_\infty} m_a \cdot M(a, c, b^2x^2/y_0^2) = I(x) = \text{a prescribed function}. \tag{15c}$$

This relation allows us to compute the m_{s_n} . We set $Z \equiv b^2x^2/y_0^2$, multiply Eq. (15c) by $[Z^{c-1} \cdot e^{-Z} \cdot M(s_n, c, Z) \cdot dZ]$, integrate from $Z = 0$ to $Z = b^2$, and use the orthogonality condition 10 to obtain

$$m_{s_n} = \frac{\displaystyle\int_0^{b^2} I((Zy_0^2/b^2)^{1/2}) \cdot Z^{c-1} \cdot e^{-Z} \cdot M(s_n, c, Z) \cdot dZ}{\displaystyle\int_0^{b^2} M(s_n, c, Z) \cdot Z^{c-1} \cdot e^{-Z} \cdot M(s_n, c, Z) \cdot dZ}. \tag{16}$$

With the coefficients m_{s_n} known from Eq. (16), the solution $u(x, t)$ is given by Eq. (15a). This solution has a domain $t > 0$, $0 \leqq x \leqq y(t)$, all material having been ablated (evaporated) for $y(t) < x < y(0) = y_0$.

3. The melting problem with inward motion. Suppose that a slab, cylinder, or sphere is being symmetrically heated from the outside and that at zero time the surface temperature is equal to the melting temperature $u = 0$. For $t > 0$ there will exist both an outer liquid region and a solid inner region. The temperature will be zero along the moving phase boundary $x = y(t)$, but there will be a discontinuity in the temperature gradient at $x = y(t)$. This discontinuity occurs because of the existence of a step change in thermal properties at the phase change temperature and because of the existence of a latent heat of phase change. The solution for the temperature distribution in the outer liquid region will be found by determining the solution for the inner solid region and then continuing that solution *across* the moving phase boundary.

Let $u(x, t)$ represent the dimensionless temperature in the solid inner region

$0 \leqq x \leqq y(t)$, $t > 0$. Let conditions (14) be imposed. Then $u(x, t)$ is given by Eq. (15a). We rewrite that equation in the form

$$u(x, t) = \sum_{a=s_0}^{a=s_\infty} m_a \cdot M(a, c, z_0 x^2/y^2) \cdot (y^2/y_0^2)^{-a}, \tag{17a}$$

where

$$z_0 \equiv b^2 = -\tfrac{1}{2} y_0 y_1 = -\tfrac{1}{2} \cdot Y(0) \cdot Y_T(0)/D, \tag{17b}$$

with

$$y^2/y_0^2 = 1 + 2y_1 t/y_0 = 1 + 2 \cdot T \cdot Y_T(0)/Y(0) = Y^2(T)/Y^2(0). \tag{17c}$$

Here $Y(T)$ is the actual (dimensional) location of the moving phase boundary. To conform with definitions (3c), we must have

$$y(t) \equiv Y(T)/X^*, \quad \text{and} \quad t \equiv TD/(X^*)^2, \tag{17d}$$

where X^* is an arbitrary reference length. Also, $Y(0)$ and $Y_T(0)$ are the arbitrary initial location and initial velocity of the moving phase boundary. Finally, as before, the s_n and m_{s_n} are determined from Eqs. (15b) and (16) but with b^2 replaced by the new constant z_0.

The density R is assumed not to vary between the solid and liquid phases. If there is no absorption of latent heat at the moving phase boundary and no change in the thermal diffusivity, then Eq. (17a) constitutes the complete solution for $y(0) \geqq x \geqq 0$. Suppose temporarily that there is no absorption of latent heat at the moving phase boundary. Then the heat flow rate is invariant across the moving boundary $x = y(t)$. But suppose that the thermal diffusivity changes stepwise from D in the inner region $y(t) > x > 0$ to D'' in the outer region $y(0) > x > y(t)$. Then the temperature distribution $u''(x, t)$ in the outer region must satisfy the following heat flux and temperature conditions at the moving phase boundary:

$$D'' \cdot u_x''(y(t), t) = D \cdot u_x(y(t), t), \tag{18a}$$

and

$$u''(y(t), t) = 0, \quad \text{with} \quad y^2/y_0^2 = 1 + 2 \cdot T \cdot Y_T(0)/Y(0). \tag{18b}$$

We now assume that $u''(x, t)$ is given by Eq. (11a), with a modification made to the variable z to indicate that $u''(x, t)$ is to be used for a material with diffusivity D''. We use condition (18b) to show that the equation used for $u''(x, t)$ must reduce to the form

$$u''(x, t) = \sum_{a=s_0''}^{a=s_\infty''} \left\{ m_a'' \cdot \left[\frac{M(a, c, z_0'' x^2/y^2)}{M(a, c, z_0'')} - \frac{W(a, c, z_0'' x^2/y^2)}{W(a, c, z_0'')} \right] \cdot (y^2/y_0^2)^{-a} \right\}, \tag{18c}$$

where

$$z_0'' \equiv (z_0) \cdot (D/D'') = -\tfrac{1}{2} \cdot Y(0) \cdot Y_T(0)/D''; \qquad m_a'' = \text{constant}. \tag{18d}$$

Substitution of Eqs. (17a) and (18c) into the boundary condition (18a) yields

$$s_n'' = s_n \qquad \text{for all } n \tag{18e}$$

and

$$m_{s_n} \cdot M_z(s_n, c, z_0) = m_{s_n}'' \cdot \left[\frac{M_z(s_n, c, z_0'')}{M(s_n, c, z_0'')} - \frac{W_z(s_n, c, z_0'')}{W(s_n, c, z_0'')} \right]. \tag{18f}$$

Equation (18f) is an equation for the determination of the m_{s_n}'', everything else being known. Equations (18c, d, e, f) completely determine $u''(x, t)$. The function $u''(x, t)$ is applicable to the outer liquid region and has a domain $y(t) < x < y(0)$.

Suppose now that latent heat is absorbed at the moving phase boundary at a rate $|y_t(t)|$. This heat must be delivered to the phase boundary by conduction through the outer liquid region. It is therefore necessary to augment the function $u''(x, t)$ by a solution $u'(x, t)$ of the heat equation for which $u'(y(t), t) = 0$ and for which $u_x'(y(t), t) = -y_t(t)$.* The solution $u'(x, t)$ can be obtained by setting $v^* = 1$ in Eq. (12a). The result is

$$u'(x, t) = 2 \cdot b^{k+1} \cdot \exp [-b^2] \cdot \int_{Z=b}^{Z=bx/y} Z^{-k} \cdot \exp [+Z^2] \cdot dZ, \tag{19a}$$

where *here*

$$b^2 \equiv z_0'' = (z_0) \cdot (D/D'') = -\tfrac{1}{2} \cdot Y(0) \cdot Y_T(0)/D''. \tag{19b}$$

For the melting problem, the complete solution to the case $yy_t = y_0 y_1$ is given by $u(x, t)$ for $y(t) \geqq x \geqq 0$ and by $[u''(x, t) + u'(x, t)]$ for $y(0) \geqq x \geqq y(t)$. Note that we did not have the freedom to impose a boundary condition at $x = y(0)$ because we required that the motion of the phase boundary be given by Eq. (17c).

References

1. Milton Abramowitz and Irene A. Stegun, Ed., *Handbook of mathematical functions*, U. S. National Bureau of Standards, Applied Mathematics Series #55, Washington, D. C., June 1964. Second printing, with corrections, Nov. 1964
2. H. S. Carslaw and J. C. Jaeger, *Conduction of heat in solids*, 2nd Ed. Oxford Univ. Press, London, 1959
3. Ruel V. Churchill, *Operational mathematics*, 2nd Ed. McGraw-Hill, New York, 1958; p. 259 and chapter 9
4. F. C. Frank, *Radially symmetric phase growth controlled by diffusion*, Proc. Roy. Soc. Ser. A **201**, 586–599 (1950)
5. A. Huber, *On the propagation of the melting boundary in a linear conductor* (Über das Fortschreiten der Schmelzgrenze in einen linearen Leiter), Z. Angew. Math. Mech. **19**, 1–21 (1939). (In German). (Cited by Carslaw and Jaeger [2] and Frank [4])
6. Lamé and Clapeyron, *Memorandum on the solidification by freezing of a liquid sphere*, (Memoire sur la solidification par refroidissement d'un globe liquide). Ann. Chimie et Physique, **47**, 250–256 (1831). (In French)
7. David Langford, *A closed form solution for the constant velocity solidification of spheres initially at the fusion temperature*, British J. Applied Physics **17**(2), 286 (Feb. 1966)
8. David Langford, *The freezing of spheres*, Int. J. Heat and Mass Transfer, **9**(8), 827–828 (August 1966)
9. David Langford, *Stefan's melting problem*, Doctoral Dissertation, Rensselaer Polytechnic Institute, Troy, N. Y., January 1965
10. G. A. Martynov, *Solution of the inverse Stefan problem in the case of spherical symmetry*, Soviet Physics—Technical Physics, **5**(2), 215–218 (Aug. 1960). Translated from Zhurnal Tekhnicheskoi Fiziki, **30**(2), 239–241 (Feb. 1960)
11. D. V. Redozubov, *The solution of certain types of linear thermal problems in limited and semiinfinite regions with motion of the boundary according to a $\beta(t)^{1/2}$ law*, Soviet Physics—Technical Physics, **7**(5), 459–461 (Nov. 1962). Translated from Zhurnal Tekhnicheskoi Fiziki, **32**(5), 632–637 (May 1962)
12. J. Stefan, *On the theory of ice formation, especially on ice formation in polar seas*, (Über die Theorie der Eisbildung, insbesondere über die Eisbildung im Polarmeere). Sitzungsberichte der Kaiserlichen Akademie Wiss. Wien., Math.-naturwiss. Kl, **98**(2a), 965–983 (1890). (In German)

*The condition on $u_x'(y(t), t)$ is a consequence of the boundary conditions (2b) and (18a) and of the use of Eq. (3b) to write $u'(x, t)$ in dimensionless form.