ON GÜNTHER'S STRESS FUNCTIONS FOR COUPLE STRESSES*

BY

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1. Introduction. In the absence of body forces and body couples, the stress equations of equilibrium for a continuum which can support couple stresses[1] may be written as

$$\begin{aligned} t_{ij,i} &= 0, \\ m_{ij,i} + \epsilon_{jkl} t_{kl} &= 0 \end{aligned} \tag{1.1}$$

when referred to rectangular Cartesian coordinates.[2] Günther [3] has observed that a solution of Eqs. (1.1) is provided by

$$\begin{aligned} t_{ij} &= \epsilon_{ipq} F_{qj,p} , \\ m_{ij} &= \epsilon_{ipq} G_{qj,p} + \delta_{ij} F_{pp} - F_{ji} , \end{aligned} \tag{1.2}$$

where the tensors F_{ij} and G_{ij} are arbitrary.[3] The stress field defined by Eqs. (1.2) will be referred to as *Günther's solution or Günther's representation,* and the tensor fields F_{ij} and G_{ij} will be called *Günther's stress functions.*

It was pointed out in [4] that Günther's solution is generally *incomplete,* i.e., there exist solutions of Eqs. (1.1) which cannot be represented by Eqs. (1.2). Several complete solutions were given in [4], and a simpler complete solution was given in [5]. However, all of these solutions are considerably more complex than Günther's solution in that they involve more scalar stress functions and higher order derivatives.

Because of its appealing simplicity, it is natural to ask what class of stress fields can be represented by Günther's solution. A more compelling reason for such a question is that Günther [3] and, more recently, Misicu [6] have made Günther's representation the basis of dislocation theories. These theories are left in doubt until it is known that Günther's stress functions can represent stress fields of sufficient generality. It is the purpose of the present paper to answer the above question.

In Sec. 2 two general representation theorems for (sufficiently smooth second-order) tensor fields are proved. The first of these states that any tensor field can be written as the curl of another tensor field plus the gradient of a vector field. The second theorem states that any tensor field with zero total flux across every closed surface in the region involved can be written as the curl of another tensor field. In analogy with classical theorems of vector analysis, these results may be called the *Stokes–Helmholtz resolution*

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[1] A Cosserat [1] continuum is an example of such a material. A modern treatment of the concept of couple stresses has been given by Truesdell and Toupin [2].

[2] We employ the usual indicial notation of Cartesian tensor analysis. Latin subscripts have the range (1, 2, 3), and summation over repeated subscripts is implied. Subscripts preceded by a comma indicate differentiation with respect to the corresponding Cartesian coordinate. Kronecker's delta and the alternating symbol are denoted by δ_{ij} and ϵ_{ijk}, respectively.

[3] Here and in the sequel any obvious smoothness requirements will not be stated.

and the *theorem of the tensor potential,* respectively. It will be clear from our proofs that theorems like these hold for tensor fields of all orders except zero. In the second-order case, the Stokes–Helmholtz resolution was given recently by Mindlin [7]; and the tensor potential theorem was inferred from the vector version by Gurtin [8]. Here we choose to give similar proofs based on significantly different smoothness hypotheses.

In Sec. 3 the notions of equilibrated and totally self-equilibrated stress fields are reviewed. In Sec. 4 it is shown that Günther's solution can at most represent totally self-equilibrated solutions of Eqs. (1.1). The theorem of the tensor potential is then used to prove that every totally self-equilibrated stress field admits Günther's representation. Finally, in Sec. 5 the Stokes–Helmholtz resolution is used to introduce a solution of Eqs. (1.1) which is complete even if the stresses are not totally self-equilibrated.

The results of this paper are analogous to certain theorems concerning the Beltrami stress functions for nonpolar continuum mechanics. In fact Gurtin's [8] definitive work on Beltrami's solution was used as a guide in carrying out the research presented here. One exception is that the proof of the tensor potential theorem is patterned after Stevenson's [9] proof of the vector potential theorem as was the author's [10] completeness proof for the Beltrami representation. It is interesting to note that because the stress tensors being represented are not required to be symmetric, the theorems on Günther's stress functions are more readily obtained than are those on Beltrami's stress functions.

2. Some tensor representation theorems. For the remainder of the paper R will denote a bounded open region of three-dimensional Euclidean space. The boundary of R is ∂R, and the unit outward normal to ∂R is $\mathbf{n}$. We write $f \in C_\lambda^N(R)$ $\{$or $f \in C_\lambda^N(R + \partial R)\}$ if and only if f is a real-valued function continuous and N times continuously differentiable on R $\{$or $R + \partial R\}$ whose Nth-order derivatives are Hölder continuous with exponent $\lambda < 1$ on R $\{$or $R + \partial R\}$.

THEOREM 2.1 (STOKES-HELMHOLTZ RESOLUTION). *Let* ∂R *be two times continuously differentiable. Let* $\phi_{ij} \in C_\lambda^0(R + \partial R)$ *and* $\phi_{ij} \in C_\lambda^N(R)$. *Then there exist* $\Omega_{ij} \in C_\lambda^1(R + \partial R)$, $\Omega_{ij} \in C_\lambda^{N+1}(R)$, *and* $\omega_i \in C_\lambda^1(R + \partial R)$, $\omega_i \in C_\lambda^{N+1}(R)$ *such that*

$$\phi_{ij} = \epsilon_{ipq}\Omega_{qj,p} + \omega_{j,i} \quad \textit{on} \quad R + \partial R.$$

Moreover, $\Omega_{ij,i} = 0$.

Proof. Define A_{ij} on $R + \partial R$ through

$$A_{ij}(\mathbf{x}) = -\frac{1}{4\pi}\int_R \frac{\phi_{ij}(\boldsymbol{\xi})}{|\mathbf{x} - \boldsymbol{\xi}|}\, dV_\xi\,.$$

It follows from well-known results on Newtonian potentials [11], [12] that

$$A_{ij} \in C_\lambda^2(R + \partial R),\ A_{ij} \in C_\lambda^{N+2}(R),$$

and

$$\nabla^2 A_{ij} = \phi_{ij} \quad \text{on} \quad R + \partial R. \tag{2.1}$$

Also one has the identity

$$\nabla^2 A_{ij} = -\epsilon_{ipq}\epsilon_{qmn}A_{nj,mp} + A_{pj,pi}\,, \tag{2.2}$$

as is readily verified with the aid of

$$\epsilon_{ijk}\epsilon_{ipq} = \delta_{jp}\delta_{kq} - \delta_{jq}\delta_{kp}\,. \tag{2.3}$$

The proof is completed by setting

$$\Omega_{qj} = -\epsilon_{qmn}A_{nj,m}$$

and

$$\omega_j = A_{pj,p}$$

in Eq. (2.2) and then using Eq. (2.1).

The hypotheses of Theorem 2.1 differ from those usually assumed in theorems of this type in that ∂R is required to be quite smooth and the continuity conditions on ϕ_{ij} and its derivatives are of the Hölder type. Because of these assumptions, the representation has the same smoothness properties as ϕ_{ij} and holds on ∂R as well as on R. This observation is due to Stippes [13].

THEOREM 2.2 (TENSOR POTENTIAL THEOREM). *Let ∂R consist of $n + 1$ closed surfaces S_a $(a = 0, 1, \cdots, n)$ each of which is four times continuously differentiable. Let ϕ_{ij} have the following properties:*

(i) $\phi_{ij} \in C_\lambda^2(R + \partial R), \quad \phi_{ij} \in C_\lambda^N(R)$ *with* $N \geq 2$,

(ii) $\phi_{ij,i} = 0$,

(iii) $\int_{S_a} \phi_{ij}n_i \, dA = 0 \qquad (a = 0, 1, \cdots, n)$.[4]

Then there exist $\Omega_{ij} \in C_\lambda^3(R + \partial R)$, $\Omega_{ij} \in C_\lambda^{N+1}(R)$ such that

$$\phi_{ij} = \epsilon_{ipq}\Omega_{qj,p} \quad \text{on} \quad R + \partial R.$$

Moreover, $\Omega_{ij,i} = 0$.

Proof. Number the surfaces S_a so that S_0 encloses $S_1, S_2, \cdots, S_n$. Let R_a denote the open region interior to the surface S_a $(a = 1, 2, \cdots, n)$, and let R_0 denote the open region exterior to S_0 and interior to S' where S' is any finite spherical surface which encloses S_0.

Next introduce functions $\psi_j^{(a)}$ with the properties:

$$\begin{gathered} \psi_j^{(a)} \in C_\lambda^2(R_a + S_a), \\ \nabla^2\psi_j^{(a)} = 0, \qquad (a = 1, 2, \cdots, n) \\ n_i\psi_{j,i}^{(a)} = \phi_{ij}n_i \quad \text{on} \quad S_a, \end{gathered} \tag{2.4}$$

and

$$\begin{gathered} \psi_j^{(0)} \in C_\lambda^2(R_0 + S_0 + S'), \\ \nabla^2\psi_j^{(0)} = 0, \\ n_i\psi_{j,i}^{(0)} = \phi_{ij}n_i \quad \text{on} \quad S_0, \qquad n_i\psi_{j,i}^{(0)} = 0 \quad \text{on} \quad S'.^{5} \end{gathered} \tag{2.5}$$

The existence of the solutions of these Neumann problems is guaranteed by (i), (iii), and the smoothness of S_a [12]. Of course on the open region R_a, $\psi_j^{(a)}$ will be analytic.

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[4] According to the divergence theorem, hypotheses (ii) and (iii) are equivalent to the requirement that $\int_S \phi_{ij}\, n_i \, dA = 0$ for all regular closed surfaces S contained in $R + \partial R$.

[5] Recall that on S_a **n** points out of R. On S' we take **n** to point out of R_0.

By Eqs. (2.4) and (2.5) functions $\phi_{ij}^{(a)}$ defined by $\phi_{ij}^{(a)} = \psi_{j,i}^{(a)}$ have the properties:

$$\begin{gathered}\phi_{ij}^{(a)} \varepsilon\, C_\lambda^1(R_a + S_a), \qquad \phi_{ij}^{(a)} \ \text{analytic on} \ R_a\,, \\ \phi_{ij,i}^{(a)} = 0, \qquad (a = 1, 2, \cdots, n) \\ \phi_{ij}^{(a)} n_i = \phi_{ij} n_i \quad \text{on} \quad S_a\,,\end{gathered} \tag{2.6}$$

and

$$\begin{gathered}\phi_{ij}^{(0)} \varepsilon\, C_\lambda^1(R_0 + S_0 + S'), \qquad \phi_{ij}^{(0)} \ \text{analytic on} \ R_0\,, \\ \phi_{ij,i}^{(0)} = 0, \\ \phi_{ij}^{(0)} n_i = \phi_{ij} n_i \quad \text{on} \quad S_0\,, \qquad \phi_{ij}^{(0)} n_i = 0 \quad \text{on} \quad S'.\end{gathered} \tag{2.7}$$

Finally, define B_{ij} on $R + \partial R$ through

$$B_{ij}(\mathbf{x}) = -\frac{1}{4\pi} \int_R \frac{\phi_{ij}(\xi)}{|\mathbf{x} - \xi|}\, dV_\xi - \sum_{a=0}^{n} \frac{1}{4\pi} \int_{R_a} \frac{\phi_{ij}^{(a)}(\xi)}{|\mathbf{x} - \xi|}\, dV_\xi\,. \tag{2.8}$$

Then [11], [12] $B_{ij}\ \varepsilon\ C_\lambda^4(R + \partial R)$, $B_{ij}\ \varepsilon\ C_\lambda^{N+2}(R)$, and

$$\nabla^2 B_{ij} = \phi_{ij} \quad \text{on} \quad R + \partial R. \tag{2.9}$$

Again we have the identity expressed by Eq. (2.2), i.e.,

$$\nabla^2 B_{ij} = -\epsilon_{ipq}\epsilon_{qmn} B_{nj,mp} + B_{pj,pi}\,. \tag{2.10}$$

We assert that

$$B_{pj,p} = 0. \tag{2.11}$$

Granting this for the moment, we set

$$\Omega_{qj} = -\epsilon_{qmn} B_{nj,m} \tag{2.12}$$

and obtain from Eqs. (2.9)–(2.12) that

$$\phi_{ij} = \epsilon_{ipq} \Omega_{qj,p}\,.$$

Furthermore, it follows from Eq. (2.12) that $\Omega_{ij}\ \varepsilon\ C_\lambda^3(R + \partial R)$, $\Omega_{ij}\ \varepsilon\ C_\lambda^{N+1}(R)$, and $\Omega_{ij,i} = 0$.

In order to show that $B_{ij,i} = 0$, we note from Eq. (2.8) and integration by parts [11] that

$$\begin{aligned}-4\pi B_{ij,i}(\mathbf{x}) = {} & \int_R \frac{\phi_{ij,i}(\xi)}{|\mathbf{x} - \xi|}\, dV_\xi - \sum_{a=0}^{n} \int_{S_a} \frac{\phi_{ij}(\xi) n_i(\xi)}{|\mathbf{x} - \xi|}\, dA_\xi + \sum_{a=0}^{n} \int_{R_a} \frac{\phi_{ij,i}^{(a)}(\xi)}{|\mathbf{x} - \xi|}\, dV_\xi \\ & + \sum_{a=0}^{n} \int_{S_a} \frac{\phi_{ij}^{(a)}(\xi) n_i(\xi)}{|\mathbf{x} - \xi|}\, dA_\xi - \int_{S'} \frac{\phi_{ij}^{(0)}(\xi) n_i(\xi)}{|\mathbf{x} - \xi|}\, dA_\xi\,.\end{aligned} \tag{2.13}$$

Equations (2.13), (2.6), (2.7), and the hypothesis that $\phi_{ij,i} = 0$ imply that $B_{ij,i} = 0$. This completes the proof.

3. Equilibrated and totally self-equilibrated stress fields. A stress field (t_{ij}, m_{ij}) is said to be *equilibrated* if and only if t_{ij} and m_{ij} satisfy Eqs. (1.1).

The *resultant force* and the *resultant moment* (about the origin) of an equilibrated stress field (t_{ij}, m_{ij}) on a closed surface S (contained in $R + \partial R$) are given by

$$T_j(S) = \int_S t_{ij} n_i \, dA \tag{3.1}$$

and

$$M_j(S) = \int_S m_{ij} n_i \, dA + \int_S \epsilon_{jkl} x_k t_{il} n_i \, dA, \tag{3.2}$$

respectively, where **n** is the unit outward normal to S.

An equilibrated stress field is said to be *totally self-equilibrated* if and only if

$$T_j(S) = M_j(S) = 0$$

for every closed surface S in $R + \partial R$. The following theorem shows that if ∂R consists of more than a single closed surface, then there is a considerable distinction between equilibrated and totally self-equilibrated stress fields.[6]

THEOREM 3.1. *Let ∂R consist of $n + 1$ closed surfaces S_a $(a = 0, 1, \cdots, n)$. Let (t_{ij}, m_{ij}) be an equilibrated stress field. Then (t_{ij}, m_{ij}) is totally self-equilibrated if and only if*

$$T_j(S_a) = M_j(S_a) = 0 \qquad (a = 0, 1, \cdots, n). \tag{3.3}$$

Proof. Suppose (t_{ij}, m_{ij}) is totally self-equilibrated. Then by definition Eqs. (3.3) hold. Conversely, suppose Eqs. (3.3) are satisfied. Then by Eqs. (1.1), (3.1), (3.2), (3.3), and the divergence theorem; it follows that for any closed surface S

$$T_j(S) = M_j(S) = 0,$$

i.e., (t_{ij}, m_{ij}) is totally self-equilibrated.

It is important to note that it is easy to give examples of equilibrated stress fields which are not totally self-equilibrated [8].

4. Günther's solution.

It was pointed out in Sec. 1 that Günther's solution defines an equilibrated stress field. In this section we will show that stress fields given by Günther's representation are necessarily totally self-equilibrated and that all totally self-equilibrated stress fields admit Günther's representation.

THEOREM 4.1. *Let the stress field (t_{ij}, m_{ij}) be given by*

$$t_{ij} = \epsilon_{ipq} F_{qj,p},$$

$$m_{ij} = \epsilon_{ipq} G_{qj,p} + \delta_{ij} F_{pp} - F_{ji}.$$

Then (t_{ij}, m_{ij}) is totally self-equilibrated.

Proof. Let S be any closed surface in $R + \partial R$. Then by Eq. (3.1)

$$T_j(S) = \int_S \epsilon_{ipq} F_{qj,p} n_i \, dA. \tag{4.1}$$

For each fixed j, the right hand side of Eq. (4.1) is the integral of the normal component of the curl of a vector field over the closed surface S. Hence by Stokes' theorem, $T_j(S) = 0$.

From Eq. (3.2)

$$M_j(S) = \int_S \epsilon_{ipq} G_{qj,p} n_i \, dA + \int_S (\delta_{ij} F_{pp} - F_{ji}) n_i \, dA + \int_S \epsilon_{jkl} \epsilon_{ipq} x_k F_{ql,p} n_i \, dA. \tag{4.2}$$

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[6] Of course if ∂R is a single closed surface, then every equilibrated stress field is necessarily totally self-equilibrated.

Using the identity

$$x_k F_{ql,p} = (x_k F_{ql})_{,p} - \delta_{kp} F_{ql} ,$$

Eq. (2.3), and Stokes' theorem; we obtain

$$\int_S \epsilon_{jkl} \epsilon_{ipq} x_k F_{ql,p} n_i \, dA = -\int_S (\delta_{ij} F_{pp} - F_{ji}) n_i \, dA. \tag{4.3}$$

Equations (4.2), (4.3), and Stokes' theorem imply that $M_j(S) = 0$. Therefore (t_{ij} , m_{ij}) is totally self-equilibrated and the theorem is proved.

THEOREM 4.2 (COMPLETENESS OF GÜNTHER'S REPRESENTATION). *Let ∂R satisfy the hypotheses of Theorem 2.2. Let the stress field (t_{ij} , m_{ij}) be totally self-equilibrated and meet the conditions*:

$$t_{ij} \in C^2_\lambda(R + \partial R), \qquad t_{ij} \in C^N_\lambda(R),$$

$$m_{ij} \in C^2_\lambda(R + \partial R), \qquad m_{ij} \in C^N_\lambda(R)$$

where $N \geq 2$. Then there exist $F_{ij} \in C^3_\lambda(R + \partial R)$, $F_{ij} \in C^{N+1}_\lambda(R)$, and $G_{ij} \in C^3_\lambda(R + \partial R)$, $G_{ij} \in C^{N+1}_\lambda(R)$ such that on $R + \partial R$

$$t_{ij} = \epsilon_{ipq} F_{qj,p} ,$$

$$m_{ij} = \epsilon_{ipq} G_{qj,p} + \delta_{ij} F_{pp} - F_{ji} .$$

Proof. By Theorem 3.1 we can apply Theorem 2.2 to t_{ij} . Thus there exist $F_{ij} \in C^3_\lambda(R + \partial R)$, $F_{ij} \in C^{N+1}_\lambda(R)$ such that

$$t_{ij} = \epsilon_{ipq} F_{qj,p} . \tag{4.4}$$

Next consider

$$H_{ij} = m_{ij} - \delta_{ij} F_{pp} + F_{ji} . \tag{4.5}$$

Clearly $H_{ij} \in C^2_\lambda(R + \partial R)$ and $H_{ij} \in C^N_\lambda(R)$. Let S be any regular closed surface in $R + \partial R$. Then Eqs. (4.5), (4.4), (4.3), (3.2), and the assumption that (t_{ij} , m_{ij}) is totally self-equilibrated yield

$$\int_S H_{ij} n_i \, dA = M_j(S) = 0.$$

Hence Theorem 2.2 can be applied to H_{ij} . Thus there exist $G_{ij} \in C^3_\lambda(R + \partial R)$, $G_{ij} \in C^{N+1}_\lambda(R)$ such that

$$H_{ij} = \epsilon_{ipq} G_{qj,p} ,$$

or by Eq. (4.5)

$$m_{ij} = \epsilon_{ipq} G_{qj,p} + \delta_{ij} F_{pp} - F_{ji} .$$

This completes the proof.

5. A generalization of Günther's solution. If the stress field (t_{ij}, m_{ij}) is not totally self-equilibrated, it is clear from the previous section that Günther's solution is not complete. In this section we will use the Stokes–Helmholtz resolution to give a suitable generalization of Günther's representation.

Here the inclusion of body forces and couples will present no difficulties, and accordingly we write the equilibrium equations as

$$\begin{aligned} t_{ij,i} + b_j &= 0, \\ m_{ij,i} + \epsilon_{jkl}t_{kl} + c_j &= 0. \end{aligned} \tag{5.1}$$

In Eqs. (5.1) b_j is the body force per unit volume and c_j is the body couple per unit volume.

The following theorem, which provides a solution of Eqs. (5.1), may be confirmed by direct substitution.

THEOREM 5.1. *Let f_j and g_j satisfy*

$$\nabla^2 f_j = -b_j , \qquad \nabla^2 g_j + \epsilon_{jkl} f_{l,k} = -c_j .$$

Define the stress field (t_{ij}, m_{ij}) through

$$t_{ij} = \epsilon_{ipq} F_{qj,p} + f_{j,i} ,$$

$$m_{ij} = \epsilon_{ipq} G_{qj,p} + \delta_{ij} F_{pp} - F_{ji} + g_{j,i} ,$$

where F_{ij} and G_{ij} are arbitrary, then (t_{ij}, m_{ij}) satisfies Eqs. (5.1).

The next theorem shows that this solution, which may be called the *generalized Günther representation*, is always complete.

THEOREM 5.2. (COMPLETENESS OF THE GENERALIZED GÜNTHER REPRESENTATION). *Let t_{ij} and m_{ij} meet the conditions:*

$$t_{ij} \in C_\lambda^0(R + \partial R), \qquad t_{ij} \in C_\lambda^N(R),$$

$$m_{ij} \in C_\lambda^0(R + \partial R), \qquad m_{ij} \in C_\lambda^N(R).$$

Then there exist F_{ij}, f_i, G_{ij}, and g_i each in the classes $C_\lambda^1(R + \partial R)$ and $C_\lambda^{N+1}(R)$ such that on $R + \partial R$

$$t_{ij} = \epsilon_{ipq} F_{qj,p} + f_{j,i} ,$$

$$m_{ij} = \epsilon_{ipq} G_{qj,p} + \delta_{ij} F_{pp} - F_{ji} + g_{j,i} .$$

Furthermore, if $N \geq 1$ and t_{ij} and m_{ij} satisfy Eqs. (5.1), *then*

$$\nabla^2 f_j = -b_j , \qquad \nabla^2 g_j + \epsilon_{jkl} f_{l,k} = -c_j .$$

Proof. By Theorem 2.1 there exist $F_{ij} \in C_\lambda^1(R + \partial R)$, $F_{ij} \in C_\lambda^{N+1}(R)$, and $f_i \in C_\lambda^1(R + \partial R)$, $f_i \in C_\lambda^{N+1}(R)$ such that

$$t_{ij} = \epsilon_{ipq} F_{qj,p} + f_{j,i} .$$

Again by Theorem 2.1 there exist $G_{ij} \in C_\lambda^1(R + \partial R)$, $G_{ij} \in C_\lambda^{N+1}(R)$, and $g_i \in C_\lambda^1(R + \partial R)$, $g_i \in C_\lambda^{N+1}(R)$ such that

$$m_{ij} - \delta_{ij} F_{pp} + F_{ji} = \epsilon_{ipq} G_{qj,p} + g_{j,i} .$$

The proof is completed by substituting these representations into Eqs. (5.1).

Note Added in Proof. In correspondence received after this paper had been submitted for publication, Professor Schaefer pointed out that he had already given (in a lecture delivered in September, 1965 in Augustów, Poland) the following complete solution to Eqs. (5.1):

$$t_{ij} = \epsilon_{ipq}F_{qj,p} + f_{j,i} ,$$

$$m_{ij} = \epsilon_{ipq}G_{qj,p} + \delta_{ij}F_{pp} - F_{ji} + \epsilon_{ijp}f_p + g_{j,i} ,$$

where

$$\nabla^2 f_i = -b_i , \qquad \nabla^2 g_i = -c_i .$$

The completeness of Schaefer's solution may be established by applying Theorem 2.1 to t_{ij} and $m_{ij} - \delta_{ij}F_{pp} + F_{ji} - \epsilon_{ijp}f_p$.

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