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COUPLED PAIRS OF DUAL INTEGRAL EQUATIONS*

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Abstract. Using the operator techniques as introduced by Erdélyi and Sneddon, solutions are obtained for two simultaneous pairs of dual integral equations of the Titchmarsh type. It is found that use of these operators exposes the important features of analysis and allows the solutions to be expressed in a compact form. The methods of Copson and Lowengrub and Sneddon form the basis for the solutions derived.

1. Introduction. Many of the problems of mathematical physics can be reduced to a mixed boundary value problem in potential theory. A recent book by Sneddon [1] discusses the methods of solution to various mixed boundary value problems in potential theory. For certain boundary value problems in the theory of elasticity a solution can be obtained by reducing the problem to a set of dual integral equations which are subsequently solved. Westmann [2], [3] has shown that for asymmetric shear loading in a circular region on an elastic half-space (e.g., the case of a penny-shaped crack opened by shearing tractions on its surfaces) the problem is mathematically equivalent to a special case of the simultaneous pairs of dual integral equations given in the following form:

$$\int_0^\infty [a\psi_1(\xi) + \psi_2(\xi)]J_{\nu+2}(\xi r)\, d\xi = f_1(r) \qquad (1 < r < \infty),$$

$$\int_0^\infty \xi^{2\alpha}[b\psi_1(\xi) + \psi_2(\xi)]J_{\nu+2}(\xi r)\, d\xi = f_2(r) \qquad (0 < r < 1),$$

$$\int_0^\infty [c\psi_1(\xi) + \psi_2(\xi)]J_{\nu}(\xi r)\, d\xi = f_3(r) \qquad (1 < r < \infty),$$

$$\int_0^\infty \xi^{2\alpha}[\psi_1(\xi) + \psi_2(\xi)]J_{\nu}(\xi r)\, d\xi = f_4(r) \qquad (0 < r < 1).$$

The solution to this set of equations was obtained by using the techniques of Copson [4] and Lowengrub and Sneddon [5].

The purpose here is to obtain a solution for certain simultaneous pairs of dual integral equations[1] by using the operator notation of Erdélyi and Sneddon [8] and Sneddon [9] in conjunction with Westmann's method. The Erdélyi-Kober and Hankel operators are defined as follows:

$$I_{\eta,\alpha}f(x) = 2x^{-2\alpha-2\eta}/\Gamma(\alpha) \int_0^x (x^2 - u^2)^{\alpha-1}u^{2\eta+1}f(u)\, du \qquad (\alpha > 0) \tag{1}$$

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[1]A solution given by Erdogan and Bahar [6] considers general systems of simultaneous dual integral equations by using a generalization of a method developed by Tranter [7].

$$= x^{-2\alpha-2\eta-1}\mathfrak{D}_x^n x^{2n+2\alpha+2\eta+1} I_{\eta,\alpha+n} f(x) \qquad (\alpha + n > 0), \tag{2}$$

$$K_{\eta,\alpha} f(x) = 2x^{2\eta}/\Gamma(\alpha) \int_x^\infty (u^2 - x^2)^{\alpha-1} u^{-2\alpha-2\eta+1} f(u)\, du \qquad (\alpha > 0) \tag{3}$$

$$= (-1)^n x^{2\eta-1}\mathfrak{D}_x^n x^{2n-2\eta+1} K_{\eta-n,\alpha+n} f(x) \qquad (\alpha + n > 0) \tag{4}$$

where $2\mathfrak{D}_x = (d/dx)x^{-1}$. The Hankel operator is

$$S_{\eta,\alpha} f(x) = 2^\alpha x^{-\alpha} \int_0^\infty t^{1-\alpha} J_{2\eta+\alpha}(xt) f(t)\, dt. \tag{5}$$

Some useful relations for these operators are

$$S_{\eta+\alpha,\beta} S_{\eta,\alpha} = I_{\eta,\alpha+\beta}\ , \tag{6}$$

$$S_{\eta,\alpha} S_{\eta+\alpha,\beta} = K_{\eta,\alpha+\beta}\ , \tag{7}$$

$$S_{\eta+\alpha,\beta} I_{\eta,\alpha} = S_{\eta,\alpha+\beta}\ , \tag{8}$$

$$S_{\eta,\alpha} K_{\eta+\alpha,\beta} = S_{\eta,\alpha+\beta}\ . \tag{9}$$

The inverse operators are

$$I_{\eta,\alpha}^{-1} = I_{\eta+\alpha,-\alpha}\ , \tag{10}$$

$$K_{\eta,\alpha}^{-1} = K_{\eta+\alpha,-\alpha}\ , \tag{11}$$

and

$$I_{\eta,0} f(x) = f(x), \tag{12}$$

$$K_{\eta,0} f(x) = f(x). \tag{13}$$

2. Coupled pairs of dual integral equations. The set of equations whose solution is sought is the following:

$$S_{\frac{1}{2}\mu-\alpha,2\alpha}(a\psi_1 + \psi_2) = f(x), \qquad x \in I_1\ , \tag{14}$$

$$S_{\frac{1}{2}\mu+n-\alpha,2\alpha}(\psi_1 + \psi_2) = g(x), \tag{15}$$

$$S_{\frac{1}{2}\nu-\beta,2\beta}\psi_1 = 0, \qquad x \in I_2\ . \tag{16}$$

$$S_{\frac{1}{2}\nu+n-\beta,2\beta}\psi_2 = 0, \tag{17}$$

where n is an integer and $I_1 = \{x\colon 0 \le x \le 1\}$, $I_2 = \{x\colon x > 1\}$. The analogous problem when the domain of definition for the integral equations is reversed will also be solved. To find the solution to Eqs. (14)–(17) the following trial solution analogous to Westmann is assumed:

$$\psi_1 = S_{\frac{1}{2}\nu+\beta,\frac{1}{2}(\mu-\nu)-\alpha-\beta} h_1 + S_{\frac{1}{2}\nu+\beta+n,\frac{1}{2}(\mu-\nu)-\alpha-\beta} h_2\ , \tag{18}$$

$$\psi_2 = S_{\frac{1}{2}\nu+\beta,\frac{1}{2}(\mu-\nu)-\alpha-\beta} h_3 + S_{\frac{1}{2}\nu+\beta+n,\frac{1}{2}(\mu-\nu)-\alpha-\beta} h_4\ . \tag{19}$$

If Eq. (18) is substituted into (16), and, with the help of Eqs. (8) and (9), use is made of the result

$$\begin{aligned} S_{\frac{1}{2}\nu+n+\beta,\frac{1}{2}(\mu-\nu)-\alpha-\beta} &= S_{\frac{1}{2}\nu+\beta,\frac{1}{2}(\mu-\nu)-\alpha-\beta} K_{\frac{1}{2}\mu-\alpha,n} I_{\frac{1}{2}\nu+\beta+n,-n} \\ &= S_{\frac{1}{2}\nu+\beta,\frac{1}{2}(\mu-\nu)-\alpha-\beta} I_{\frac{1}{2}\nu+\beta+n,-n} K_{\frac{1}{2}\mu-\alpha,n} \end{aligned} \tag{20}$$

then Eq. (16) is indentically satisfied provided that $h_1(x)$, $h_2(x) \neq 0$ only when $x \in I_1$. If Eq. (19) is placed into (17), then $h_4(x)$ will automatically satisfy (17) provided that

$h_4(x) \neq 0$ when $x \in I_1$. The remaining term of Eq. (17) may be written, analogously to (20),

$$S_{\frac{1}{2}\nu+n-\beta,2\beta}S_{\frac{1}{2}\nu+\beta,\frac{1}{2}(\mu-\nu)-\alpha-\beta}h_3 = S_{\frac{1}{2}\nu+n-\beta,2\beta}S_{\frac{1}{2}\nu+n+\beta,\frac{1}{2}(\mu-\nu)-\alpha-\beta}I_{\frac{1}{2}\nu+\beta,n}K_{\frac{1}{2}\mu-\alpha+n,-n}h_3 . \tag{21}$$

Hence, Eq. (17) will be satisfied only if $h_3(x) \equiv 0$.

Equations (18) and (19) are now substituted into (14) and (15) and, with the help of (20), are written in the form

$$S_{\frac{1}{2}\mu-\alpha,2\alpha}S_{\frac{1}{2}\nu+\beta,\frac{1}{2}(\mu-\nu)-\alpha-\beta}[ah_1 + I_{\frac{1}{2}\nu+\beta+n,-n}K_{\frac{1}{2}\mu-\alpha,n}(ah_2 + h_4)] = f(x), \tag{22}$$

$$S_{\frac{1}{2}\mu+n-\alpha,2\alpha}S_{\frac{1}{2}\nu+n+\beta,\frac{1}{2}(\mu-\nu)-\alpha-\beta}[K_{\frac{1}{2}\mu-\alpha+n,-n}I_{\frac{1}{2}\nu+\beta,n}(h_1) + h_2 + h_4] = g(x). \tag{23}$$

Taking

$$h_4 = -ah_2 \tag{24}$$

Eq. (22) becomes

$$aI_{\frac{1}{2}\nu+\beta,\frac{1}{2}(\mu-\nu)+\alpha-\beta}h_1 = f(x) \tag{25}$$

which may be solved by evaluation of

$$h_1(x) = a^{-1}I^{-1}_{\frac{1}{2}\nu+\beta,\frac{1}{2}(\mu-\nu)+\alpha-\beta}f(x). \tag{26}$$

Using the result of (26), Eq. (23) is solved for $h_2(x)$ in the following form:

$$h_2 = (1 - a)^{-1}[-K_{\frac{1}{2}\mu-\alpha+n,-n}I_{\frac{1}{2}\nu+\beta,n}h_1 + I^{-1}_{\frac{1}{2}\nu+\beta+n,\frac{1}{2}(\mu-\nu)+\alpha-\beta}g]. \tag{27}$$

The solution for the case when the boundary conditions for Eqs. (14)–(17) are reversed will now be discussed. The analogous simultaneous equations become

$$S_{\frac{1}{2}\mu-\alpha,2\alpha}\psi_1 = 0, \qquad x \in I_1 , \tag{28}$$

$$S_{\frac{1}{2}\mu+n-\alpha,2\alpha}\psi_2 = 0, \tag{29}$$

$$S_{\frac{1}{2}\nu-\beta,2\beta}(\psi_1 + \psi_2) = h(x), \qquad x \in I_2 . \tag{30}$$

$$S_{\frac{1}{2}\nu+n-\beta,2\beta}(\psi_1 + b\psi_2) = j(x), \tag{31}$$

As a trial solution, ψ_1 and ψ_2 are taken in the following form:

$$\psi_1 = S_{\frac{1}{2}\nu+\beta,\frac{1}{2}(\mu-\nu)-\alpha-\beta}h_1 , \tag{32}$$

$$\psi_2 = S_{\frac{1}{2}\nu+\beta,\frac{1}{2}(\mu-\nu)-\alpha-\beta}h_2 + S_{\frac{1}{2}\nu+\beta+n,\frac{1}{2}(\mu-\nu)-\alpha-\beta}h_3 \tag{33}$$

where $h_i(x) \neq 0$ only when $x \in I_2$ ($i = 1, 2, 3$). This selection for $h_i(x)$ automatically satisfies Eqs. (28) and (29). Setting

$$h_1 = -bh_2 \tag{34}$$

and putting (32) and (33) into (31), an equation for $h_3(x)$ is found as

$$bS_{\frac{1}{2}\nu+n-\beta,2\beta}S_{\frac{1}{2}\nu+\beta+n,\frac{1}{2}(\mu-\nu)-\alpha-\beta}h_3(x) = j(x) \tag{35}$$

or, with the help of (7),

$$K_{\frac{1}{2}\nu+n-\beta,\frac{1}{2}(\mu-\nu)-\alpha+\beta}h_3 = b^{-1}j(x). \tag{36}$$

Solution for $h_3(x)$ will require evaluation of the following result:

$$h_3(x) = b^{-1}K^{-1}_{\frac{1}{2}\nu-\beta+n,\frac{1}{2}(\mu-\nu)-\alpha+\beta}j(x). \tag{37}$$

The remaining function, $h_2(x)$, is found by putting (32) and (33) into (30), and, with the result of (37), the following solution is obtained:

$$h_2 = (1 - b)^{-1}[-I_{\frac{1}{2}\nu+\beta+n,-n}K_{\frac{1}{2}\mu-\alpha,n}h_3(x) + K^{-1}_{\frac{1}{2}\nu-\beta,\frac{1}{2}(\mu-\nu)-\alpha+\beta}h(x)]. \tag{38}$$

It should be noted that the method above is not limited only to the case of two simultaneous pairs of dual integral equations. Provided that all Bessel functions differ by an even integer, higher systems of simultaneous pairs of dual integral equations may be solved and analogous results obtained. The next section will consider the evaluation of example cases of the preceding solutions.

3. Example cases. The special cases considered are for $\mu = \nu$, $\beta = 0$, and $|\alpha| < 1$. The solutions are given by Eqs. (26) and (27) and the companion Eqs. (37) and (38) which have to be evaluated, the evaluation proceeding as in Sneddon [9]. For each of the examples considered, it is necessary to specify separately whether $0 < \alpha < 1$ or $-1 < \alpha < 0$. Upon evaluation of (26), (27) the following results are obtained ($x \in I_1$):

a. $-1 < \alpha < 0$:

$$h_1(x) = \frac{2a^{-1}x^{-\nu}}{\Gamma(-\alpha)} \int_0^x (x^2 - u^2)^{-\alpha-1} u^{\nu+2\alpha+1} f(u)\, du, \tag{39}$$

$$h_2(x) = (1 - a)^{-1}\Bigg[-\frac{(-1)^n}{\Gamma(n)} 2x^{\nu-2\alpha+2n-1} \mathfrak{D}_x^n x^{-2(\nu-\alpha+n)+1} \cdot \int_0^x (x^2 - u^2)^{n-1} u^{\nu+1} h_1(u)\, du + \frac{2x^{-\nu-2n}}{\Gamma(-\alpha)} \int_0^x (x^2 - u^2)^{-\alpha-1} u^{\nu-2\alpha+2n+1} g(u)\, du \Bigg]; \tag{40}$$

b. $0 < \alpha < 1$:

$$h_1(x) = \frac{a^{-1}x^{-\nu-1}}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_0^x (x^2 - u^2)^{-\alpha} u^{\nu+2\alpha+1} f(u)\, du, \tag{41}$$

$$h_2(x) = (1 - a)^{-1}\Bigg[-\frac{(-1)^n}{\Gamma(n)} 2x^{\nu-2\alpha+2n-1} \mathfrak{D}_x^n x^{-2(\nu-\alpha+n)+1} \cdot \int_0^x (x^2 - u^2)^{n-1} u^{\nu+1} h_1(u)\, du + \frac{x^{-\nu-1-2n}}{\Gamma(1 - \alpha)} \frac{d}{dx} \int_0^x (x^2 - u^2)^{-\alpha} u^{\nu+2\alpha+2n+1} g(u)\, du \Bigg]. \tag{42}$$

For Eqs. (37) and (38) the following results are obtained ($x \in I_2$):

c. $-1 < \alpha < 0$:

$$h_3(x) = -\frac{b^{-1}}{\Gamma(1 + \alpha)} x^{\nu-2\alpha+2n-1} \frac{d}{dx} \int_x^\infty (u^2 - x^2)^{\alpha} u^{-2n-\nu+1} f(u)\, du, \tag{43}$$

$$h_2(x) = (1 - b)^{-1}\Bigg[-\frac{2x^{-\nu-1}}{\Gamma(n)} \mathfrak{D}_x^n x^{2(\nu+n-\alpha)+1} \cdot \int_x^\infty (u^2 - x^2)^{n-1} u^{-2n-\nu+2\alpha+1} h_3(u)\, du - \frac{x^{\nu-2\alpha-1}}{\Gamma(1 + \alpha)} \frac{d}{dx} \int_x^\infty (u^2 - x^2)^{\alpha} u^{-\nu+1} h(u)\, du \Bigg]; \tag{44}$$

d. $0 < \alpha < 1$:

$$h_3(x) = \frac{2b^{-1}}{\Gamma(\alpha)} x^{\nu-2\alpha+2n} \int_x^\infty (u^2 - x^2)^{\alpha-1} u^{-\nu-2n+1} j(u)\, du, \tag{45}$$

$$h_2(x) = (1 - b)^{-1}\Bigg[-\frac{2x^{-\nu-1}}{\Gamma(n)} \mathfrak{D}_x^n x^{2(\nu+n-\alpha)+1} \cdot \int_x^\infty (u^2 - x^2)^{n-1} u^{-2n-\nu+2\alpha+1} h_3(u)\, du + \frac{2x^{\nu-2\alpha}}{\Gamma(\alpha)} \int_x^\infty (u^2 - x^2)^{\alpha-1} u^{-\nu+1} h(u)\, du \Bigg]. \tag{46}$$

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