SUB- AND SUPERHARMONIC SYNCHRONIZATION IN WEAKLY NONLINEAR SYSTEMS: INTEGRAL CONSTRAINTS AND DUALITY*

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Abstract. A forced system described by the differential equation:

$$x'' + \epsilon f(x, x') + \omega_0^2 x = F \cos \omega t$$

is considered for cases where $(n/m)\omega$ is close to ω_0. (Here m and n are integers and $n/m > 1$ denotes superharmonic while $n/m < 1$ denotes subharmonics.) If the unforced system ($F = 0$) is conservative, the forced system is shown to possess an integral constraint and the solution is reduced to quadratures, even though the force adds or removes energy from the oscillations. Furthermore, the sub- and superharmonic cases where the n/m ratios are inverse are shown to be intimately related, and results for one can be deduced from the other by appropriate interchange of variables.

For systems which are nonconservative (when $F = 0$), there is a general class, including the frequently discussed Van der Pol oscillator, whose members are mathematical duals of appropriate conservative systems with added linear dissipation. Both the nonconservative system and its "conservative" dual are forced. The duality consists of an interchange of the roles of the dissipation and detuning between the systems and yields a pair of phase portraits with singularities located at identical points and orthogonal phase trajectories.

Examples for polynomial nonlinearities are given and in considerable detail for the power 5.

1. Introduction. In a previous paper [1] the exchange of energy between oscillations in weakly-nonlinear systems was investigated. Two integral constraints on the amplitude and phase variation of the oscillations of an autonomous multi-degree of freedom system were obtained. One was the anticipated constancy of the system energy. The other was related to the exchange of energy between the oscillatory modes. In this paper, we shall consider a single degree of freedom conservative system driven externally by a sinusoidal force. We first will show that an integral constraint, similar to the second one described above, exists for oscillations in this system with frequencies that are fractions (subharmonics) or multiples (superharmonics) of the driving frequency even though the external force supplies or removes energy.

Next, we will consider a class of nonconservative systems (which include the frequently-discussed Van der Pol oscillator) subject to external driving, and show that such systems are mathematical duals of driven conservative systems (such as the Duffing equation). By interchanging the roles of the damping and the detuning, the two systems

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are shown to have the same synchronous (steady state) characteristics. Furthermore, the phase plane trajectories (transient state), where the phase plane coordinates are phase and log amplitude of the subharmonic or superharmonic oscillations, of the related dual systems are orthogonal. Previous writers [2] to [6] have considered examples of the conservative and nonconservative cases separately and have not observed the duality.

Finally, a relationship between subharmonics and superharmonics in a forced conservative system is developed. By appropriate interchanges of the amplitudes, phases and frequencies of the response components (sub- or superharmonic and the oscillation at the driving frequency), a complementary solution is obtained, the subharmonic in the original becoming a superharmonic or vice versa.

2. Lagrangian formulation. Consider a conservative single-degree of freedom system possessing the Lagrangian:

$$L(x, x') = \tfrac{1}{2}(x')^2 - \tfrac{1}{2}\omega_0^2 x^2 + \epsilon l(x, x'), \qquad (\,' = d/dt), \qquad \epsilon \ll 1 \tag{1}$$

where l represents the effects of the nonlinearity.

The differential equation which we will be considering throughout this paper will be restricted to the form

$$x'' + \omega_0^2 x = -\epsilon f(x, x') + F \cos \omega t \tag{2}$$

where f is a general nonlinear function and $F \cos \omega t$ is the external driving force. Since f may not contain x'' and since $f = (d/dt)(\partial l/\partial x') - \partial l/dx$ for the conservative system, it follows that l must be linear in x' and so in fact, in the conservative case, f reduces to a function of x only. To make sections of the analysis applicable later to nonconservative systems, both arguments of f will however be retained.

We shall examine (2) for the existence of synchronized subharmonic or superharmonic oscillations; that is, for steady-state solutions which contain major components at a frequency $(n/m)\omega$. Here n and m are integers in lowest terms and $n < m$ denotes the subharmonic case. The physically interesting case is when the subharmonic or superharmonic is of order unity (rather than $O(\epsilon)$ or higher) and the analysis is focused on these. By substituting:

$$x = y - S \cos \omega t; \qquad S = \frac{F}{(\omega^2 - \omega_0^2)} = O(1) \tag{3}$$

and

$$\epsilon\gamma = (n^2/m^2)\omega^2 - \omega_0^2$$

into (2), we obtain:

$$y'' + (n^2/m^2)\omega^2 y = -\epsilon[f(y - S \cos \omega t, y' + S\omega \sin \omega t) - \gamma y]. \tag{4}$$

The solution of this nonautonomous differential equation may be represented by an asymptotic expansion, according to [7], involving two time variables:

$$y(t, \tau, \epsilon) = y_0(t, \tau) + \epsilon y_1(t, \tau) + \epsilon^2 y_2(t, \tau) + \cdots \tag{5}$$

where $\tau = \epsilon t$ is the slow time scale. The variables t and τ are treated as independent.

Further expansion of τ and ω_0^2 in ϵ is necessary in order to obtain uniformly valid second and higher order approximations. Following the analysis in [8] and [9], upon substitution of (5) into (4) and collection of terms of like order of ϵ, we obtain:

$$\epsilon^0 \text{ terms: } \partial^2 y_0/\partial t^2 + (n^2/m^2)\omega^2 y_0 = 0, \tag{6}$$

$$\begin{aligned}\epsilon \text{ terms: } &\partial^2 y_1/\partial t^2 + (n^2/m^2)\omega^2 y_1 \\ &= -f(y_0 - S\cos\omega t,\ \partial y_0/\partial t + S\omega\sin\omega t) + \gamma y_0 - 2\,\partial^2 y_0/\partial t\,\partial\tau.\end{aligned} \tag{7}$$

The solution of (6) is of the form:

$$y_0(t, \tau) = R(\tau)\cos[(n/m)\omega t + \phi(\tau)]. \tag{8}$$

This solution is then substituted into the right-hand side of (7) and the secular terms are suppressed by eliminating terms that contain sin $[(n/m)\omega t + \phi]$ and cos $[(n/m)\omega t + \phi]$:

$$2\frac{n}{m}\omega\frac{dR}{d\tau} = \left[f\left(y_0 - S\cos\omega t, \frac{\partial y_0}{\partial t} + S\omega\sin\omega t\right)\right]_{\sin[(n/m)\omega t+\phi]\text{ terms}}, \tag{9}$$

$$2\frac{n}{m}\omega R\frac{d\phi}{d\tau} = \left[f\left(y_0 - S\cos\omega t, \frac{\partial y_0}{\partial t} + S\omega\sin\omega t\right)\right]_{\cos[(n/m)+\phi]\text{ terms}} - \gamma R. \tag{10}$$

The notation $[f]\{\sin[(n/m)\omega t + \phi]; \cos[(n/m)\omega t + \phi]\}$ terms implies $(\omega/\pi m)\int_0^{2m\pi/\omega}$ $[f]\{\sin[(n/m)\omega t + \phi]; \cos[(n/m)\omega t + \phi]\}\,dt$, where τ is held constant during the integration.

The right-hand sides of Eqs. (9) and (10) are functions of R, ϕ and S, the latter being specified through (3). The simultaneous solution of (9) and (10) yields the transient behavior of y_0. The singular points of these equations represent all synchronized solutions. These points are located at the common roots of (9) and (10) or:

$$\begin{aligned}[f]_{\sin[(n/m)\omega t+\phi]} &= 0, \\ [f]_{\cos[(n/m)\omega t+\phi]} &= \gamma R.\end{aligned} \tag{11}$$

A necessary condition for synchronization is that either expression of (11) contain ϕ explicitly. Using this idea, a procedure was developed, in [8] or [9], for determining whether a given nonlinearity can produce a particular synchronized oscillation. Terms of the right hand sides of (9) or (10) which contain ϕ will be called synchronous terms.

The equation set (9) and (10) can be shown to possess an integral constraint. First, consider the average of the incremental Lagrangian over one period of the "fast" time variable:

$$l^*(R, S, \varphi) = \frac{\omega}{2\pi m}\int_0^{2m\pi/\omega} l(x, x')\,dt. \tag{12}$$

The R and ϕ are held constant for the integration since they are functions of τ.

Now let us consider the slow variation of the integral (12):

$$\begin{aligned}\frac{dl^*}{d\tau} &= \frac{\partial l^*}{\partial R}\frac{dR}{d\tau} + \frac{\partial l^*}{\partial \phi}\frac{d\phi}{d\tau}, \\ &= \frac{\omega}{2m\pi}\left\{\frac{dR}{d\tau}\int_0^{2m\pi/\omega}\left(\frac{\partial l}{\partial x}\frac{dx}{dR} + \frac{\partial l}{\partial x'}\frac{dx'}{dR}\right)dt + \frac{d\phi}{d\tau}\int_0^{2m\pi/\omega}\left(\frac{\partial l}{\partial x}\frac{dx}{d\phi} + \frac{\partial l}{\partial x'}\frac{dx'}{d\phi}\right)dt\right\}.\end{aligned} \tag{13}$$

The derivatives $\partial x/\partial R$, $\partial x/\partial\phi$, $\partial x'/\partial R$, $\partial x'/\partial\phi$ are found from

$$\begin{aligned} x &= R\cos\left(\frac{n}{m}\omega t+\phi\right) - S\cos\omega t, \\ x' &= -\frac{n}{m}\omega R\sin\left(\frac{n}{m}\omega t+\phi\right) + S\omega\sin\omega t + \epsilon(\cdots) \end{aligned} \tag{14}$$

and the expression (13), for zero order terms, becomes:

$$\begin{aligned} \frac{dl^*}{d\tau} = \frac{\omega}{2\pi m}\Biggl\{\frac{dR}{d\tau}\int_0^{2\pi m/\omega}\left[\frac{\partial l}{\partial x}\cos\left(\frac{n}{m}\omega t+\phi\right) - \frac{n}{m}\omega\frac{\partial l}{\partial x'}\sin\left(\frac{n}{m}\omega t+\phi\right)\right]dt \\ - R\frac{d\phi}{d\tau}\int_0^{2\pi m/\omega}\left[\frac{\partial l}{\partial x}\sin\left(\frac{n}{m}\omega t+\phi\right) + \frac{n}{m}\omega\frac{\partial l}{\partial x'}\cos\left(\frac{n}{m}\omega t+\phi\right)\right]dt\Biggr\}. \end{aligned} \tag{15}$$

Now let us examine the expressions (9) and (10) using the definition

$$f = \frac{d}{dt}\left(\frac{\partial l}{\partial x'}\right) - \frac{\partial l}{\partial x},$$

$$2\frac{n}{m}\omega\frac{dR}{d\tau} = \frac{\omega}{\pi m}\int_0^{2\pi m/\omega}\left[\frac{d}{dt}\left(\frac{\partial l}{dx'}\right) - \frac{\partial l}{\partial x}\right]\sin\left(\frac{n}{m}\omega t+\phi\right)dt,$$

$$2\frac{n}{m}\omega R\frac{d\phi}{d\tau} = \frac{\omega}{\pi m}\int_0^{2\pi m/\omega}\left[\frac{d}{dt}\left(\frac{\partial l}{\partial x'}\right) - \frac{\partial l}{\partial x}\right]\cos\left(\frac{n}{m}\omega t+\phi\right)dt.$$

The operation d/dt implies $\partial/\partial t + \epsilon(\partial/\partial\tau)$, and the second term is neglected since it is of a higher order. A parts integration performed on the first term of (9) or (10) yields:

$$2\frac{n}{m}\omega\frac{dR}{d\tau} = -\frac{\omega}{\pi m}\int_0^{2\pi m/\omega}\left[\frac{n}{m}\omega\frac{\partial l}{\partial x'}\cos\left(\frac{n}{m}\omega t+\phi\right) + \frac{\partial l}{\partial x}\sin\left(\frac{n}{m}\omega t+\phi\right)\right]dt, \tag{16}$$

$$2\frac{n}{m}\omega R\frac{d\phi}{d\tau} = \frac{\omega}{\pi m}\int_0^{2\pi m/\omega}\left[\frac{d}{dt}\left(\frac{\partial l}{\partial x'}\sin\left(\frac{n}{m}\omega t+\phi\right) - \frac{\partial l}{\partial x}\cos\left(\frac{n}{m}\omega t+\phi\right)\right]dt - \gamma R. \tag{17}$$

Substitution of (16) and (17) into (15) then gives:

$$\frac{dl^*}{d\tau} = -\frac{dR}{d\tau}\left(2\frac{n}{m}\omega R\frac{d\phi}{d\tau} + \frac{\gamma}{2}R\right) + R\frac{d\phi}{d\tau}\left(2\frac{n}{m}\omega\frac{dR}{d\tau}\right) = -\frac{\gamma}{2}R\frac{dR}{d\tau}. \tag{18}$$

Therefore,

$$\psi(R,\phi) = l^*(R,\phi) + \frac{\gamma}{4}R^2 = \text{constant} \tag{19}$$

is an integral constraint on the R and ϕ variables due to the conservative nature of the system. A similar integral constraint was obtained in [1] in an investigation of the free oscillations in conservative quasi-linear systems with multiple degrees of freedom. It is interesting to observe from (19) and (16) that:

$$\frac{\partial\psi}{\partial\phi} = 2\frac{n}{m}\omega R\frac{dR}{d\tau} \tag{20}$$

and from (19) and (17):

$$\frac{\partial\psi}{\partial R} = -2\frac{n}{m}\omega R\frac{d\phi}{d\tau} \tag{21}$$

Therefore, it follows that:

$$\frac{\partial}{\partial R}\left[R\,\frac{dR}{d\tau}\right] = -\frac{\partial}{\partial\phi}\left[R\,\frac{\partial\phi}{\partial\tau}\right] \tag{22}$$

The expression (22) presents a relationship between the contributions of the nonlinearity (i.e., perturbation Lagrangian) to the variations in the amplitude and phase of the subharmonic or superharmonic oscillation.

3. Duality between a system with conservative nonlinearity and one with nonconservative nonlinearity. Consider now a system described by (2) where the force $f(x, x')$ consists of a nonlinear conservative part, $g(x)$ and a linear dissipative part $\beta x'$ (i.e., a Duffing type of system). Following the procedure outlined in section 2, the amplitude and phase of the oscillation $R\cos[(n/m)\omega t + \phi]$ satisfy the following differential equations:

$$2\,\frac{n}{m}\,\omega\,\frac{dR}{d\tau} = \left[g\left(R\cos\left(\frac{n}{m}\,\omega t + \phi\right) - S\cos\omega t\right)\right]_{\sin[(n/m)\omega t+\phi]\text{ terms}} - \frac{n}{m}\,\omega\beta R, \tag{23}$$

$$2\,\frac{n}{m}\,\omega R\,\frac{d\phi}{d\tau} = \left[g\left(R\cos\left(\frac{n}{m}\,\omega t + \phi\right) - S\cos\omega t\right)\right]_{\cos[(n/m)\omega t+\phi]\text{ terms}} - \gamma R. \tag{24}$$

The subharmonic or superharmonic characteristics in the steady and transient states are determined from the trajectories of (23) and (24) in the $R - \phi$ plane, and in particular from the singular points and their stability.

Next let us consider a second system with a force $f(x, x')$ given by

$$f(x, x') = T(dg/dx)x' - \chi x'$$

where $g(x)$ is the same nonlinear function as above, but here however the term incorporating it is nonconservative. The term $\chi x'$ is a negative damping, and the system is of the Van der Pol type. The function f may be rewritten

$$f(x, x') = T\,dg/dt - \chi x'$$

where the operator (d/dt) may be interpreted again as $\partial/\partial t + \epsilon\partial/\partial\tau$, and T is a time constant. The amplitude and phase of the oscillation $R\cos[(n/m)\omega t + \phi]$ in such a system satisfy:

$$2\,\frac{n}{m}\,\omega\,\frac{dR}{d\tau} = T\left[\frac{\partial}{\partial t}\,g\left(R\cos\left(\frac{n}{m}\,\omega t + \phi\right) - S\cos\omega t\right)\right]_{\sin[(n/m)\omega t+\phi]\text{ terms}} + \frac{n}{m}\,\omega\chi R, \tag{25}$$

$$2\,\frac{n}{m}\,\omega R\,\frac{d\phi}{d\tau} = T\left[\frac{\partial}{\partial t}\,f\left(R\cos\left(\frac{n}{m}\,\omega t + \phi\right) - S\cos\omega t\right)\right]_{\cos[(n/m)\omega t+\phi]\text{ terms}} - \mu R \tag{26}$$

where $\mu = (1/\epsilon)[(n^2/m^2)\omega^2 - \omega_0^2]$ represents the detuning just as γ does in (24). A parts integration in the "fast" variable of the first term on the right-hand sides of (25) and (26) yields:

$$2\,\frac{n}{m}\,\omega\,\frac{dR}{d\tau} = -\frac{n}{m}\,\omega T\left[g\left(R\cos\left(\frac{n}{m}\,\omega t + \phi\right) - S\cos\omega t\right)\right]_{\cos[(n/m)\omega t+\phi]\text{ terms}} + \frac{n}{m}\,\omega\chi R, \tag{27}$$

$$2\,\frac{n}{m}\,\omega R\,\frac{d\phi}{d\tau} = \frac{n}{m}\,\omega T\left[g\left(R\cos\left(\frac{n}{m}\,\omega t + \phi\right) - S\cos\omega t\right)\right]_{\sin[(n/m)\omega t+\phi]\text{ terms}} - \mu R. \tag{28}$$

A comparison of (23), (24) and (27), (28) shows that if

$$\frac{\chi}{T} = \gamma, \tag{29}$$

$$[m/(n\omega)]^2\mu/T = \beta. \tag{30}$$

Then

$$[(1/R)(dR/d\phi)]_{\text{conservative}} = -[R(d\phi/dR)]_{\text{nonconservative}} . \tag{31}$$

The relationships (29) and (30) imply that the damping and detuning in one system are equivalent to the detuning and damping respectively in the other system.

The expression (31) indicates that for equivalent, or "dual" systems, integral curves (transient state) of both systems are in fact orthogonal in a phase plane whose coordinates are log R and ϕ. In the orthogonal transformation of integral curves, a saddle type of singularity remains a saddle but with a shift in orientation. A center becomes a node while foci transform into other foci or nodes.

4. Relationship between subharmonics and superharmonics for conservative systems. Consider the expression (14). If we let:

$$\begin{aligned} \omega_1 t &= \frac{n}{m}\omega t + \phi; & \phi_1 &= -\frac{m}{n}\phi, \\ R_1 &= -S; & S_1 &= -R. \end{aligned} \tag{32}$$

Then (14) becomes:

$$\begin{aligned} x &= -S_1 \cos \omega_1 t + R_1 \cos\left(\frac{m}{n}\omega_1 t + \phi_1\right), \\ x' &= +\omega_1 S_1 \sin \omega_1 t - \frac{m}{n}\omega_1 R_1 \sin\left(\frac{m}{n}\omega_1 t + \phi_1\right) + \epsilon(\cdots). \end{aligned} \tag{33}$$

The expressions (33) describe another oscillation which is complementary to the original one: superharmonic if the original is subharmonic and vice versa. According to our conventions, the frequency of the R component of x is close to the linear resonance frequency of the system. Therefore in the original oscillation $\omega \cong (m/n)\omega_0$ while in the complementary oscillation $\omega_1 \cong (n/m)\omega_0$. The force required to produce the S_1 component at the driving frequency in the complementary oscillation is given by $F_1 = S_1(\omega_1^2 - \omega_0^2)$.

Assume now that for the specified conservative nonlinearity and l_1^* is formed from (12) for the complementary oscillation. This must, of course, be identical with l^* for the original oscillation provided the arguments are altered:

$$\{l_1^*(R_1, S_1, \phi_1)\}_{m/n} = \left\{l^*\left(-S_1, -R_1, -\frac{n}{m}\phi_1\right)\right\}_{n/m} . \tag{34}$$

That is, if l^* is known for an oscillation with a given n/m, it can be found for the complementary oscillation simply by exchanging the R, S and ϕ variables according to (32). A comparison may also be made between the forms which (20) takes for the original and complementary oscillations:

$$\begin{aligned} 2\omega_0 R \frac{dR}{d\tau} &= \frac{\partial l^*}{\partial \phi}, \\ 2\omega_0 R_1 \frac{dR_1}{d\tau} &= \frac{\partial l_1^*}{\partial \phi_1} = -\frac{n}{m}\frac{\partial l^*}{\partial \phi}\bigg|_{R=-S_1,\ S=-R_1,\ \phi=-(n/m)\phi_1} . \end{aligned} \tag{35}$$

Thus the equation for R_1 can be simply derived from the R equation by the interchange of variables and multiplication by $-n/m$. The same is *not* true for equation (21), since to form $\partial l_1^*/\partial R_1$ requires $\partial l^*/\partial S$ which is different from $\partial l^*/\partial R$.

5. Example: Generalized Duffing equation. To illustrate the foregoing generalities, we will consider a particular kind of conservative nonlinearity, $f(x, x') = x^p$, where p is odd. It has been shown ([8] or [9]) that only certain subharmonics or superharmonics can be sustained by such a function (in an analysis which is limited to terms of order ϵ). The underlying concept is that for synchronization to be possible, the right sides of either (9) or (10) must contain ϕ explicitly. Without repeating details, the result for $p =$ odd is

$$\frac{m}{n} = \frac{p+1-u-2v}{u} \quad \text{where} \quad u = 1, 2, 3, \cdots p; \quad v = 0, 1, 2 \cdots \begin{cases} (p-u)/2 & \text{for } u \text{ odd}, \\ (p-u-1)/2 & \text{for } u \text{ even}. \end{cases}$$

When $p = 9$, for example, the formula yields a table:

u	m/n	u	m/n
1	9/1, 7/1, 5/1, 3/1, 1/1	6	4/6, 2/6
2	8/2, 6/2, 4/2, 2/2	7	3/7, 1/7
3	7/3, 5/3, 3/3, 1/3	8	2/8
4	6/4, 4/4, 2/4	9	1/9.
5	5/5, 3/5, 1/5		

As we would expect from the result of the previous section, for every m/n there is a corresponding n/m. Note that the ratio $\frac{6}{2}$ when reduced to lowest terms becomes $\frac{3}{1}$, which is a repeat of an earlier entry. This repetition is not without significance, and for future purposes we will note that for $p = 9$ there are two values of m (3 and 6) which give rise to a subharmonic of order $\frac{1}{3}$. Since any ratio in the table is constrained by $m + n \leq p + 1$, it follows that repetitions occur only for large p.

For $f = x^p$,

$$l' = -\frac{x^{p+1}}{p+1} \tag{36}$$

and l^* given by (12) is (see [8] for details):

$$l^*(R, S, \phi) = -R \sum_m \frac{1}{m} C_{m,n}^p(R, S) \cos m\phi - \int B(R, S)\, dR \tag{37}$$

where

$$B(R, S) = \frac{p!}{2^{p-1}} \sum_{k=0}^{(p-1)/2} \frac{R^{p-2k} S^{2k}}{\left(\frac{p+1}{2} - k\right)! \left(\frac{p-1}{2} - k\right)! \,(k!)^2} \tag{38}$$

and

$$C_{m,n}^p = \frac{p!}{2^{p-1}} \sum_{k=0}^{(p-m)/2} \frac{(-1)^{2k+1} m R^{p-2k+1} S^{2k+1}}{\left(k - \frac{n-1}{2}\right)! \left(k + \frac{n+1}{2}\right)! \left(\frac{p-m}{2} - k\right)! \left(\frac{p+m}{2} - k\right)!}, \tag{39}$$

$$k \geq (n-1)/2$$

with m = odd, or

$$C^p_{m,n} = \frac{p!}{2^{p-1}} \sum_{k=0}^{(p-m)/2} \frac{mR^{p-2k}S^{2k}}{\left(k - \frac{n}{2}\right)! \left(k + \frac{n}{2}\right)! \left(\frac{p - m + 1}{2} - k\right)! \left(\frac{p + m + 1}{2} - k\right)!}, \tag{40}$$

$$k \geq n/2$$

with m even. The summation over m in (37) means over all m relevant to the chosen subharmonic and p. (That is, $m = 3, 6$ for the example cited above.) In the coefficient $C_{m,n}$ the value of n associated with each m is the one which preserves the ratio at the specified value. If p is small the sum reduces to a single term. It is also interesting to note that if the signs of t and ϕ are both changed in the integrand of (12) the integrand itself remains unchanged. Since changing the sign of t cannot alter the value of l^* it follows that l^* must be an even function of ϕ, as (37) shows.

If linear damping, $\epsilon\beta x'$, is added to the system so that $f(x, x') = x^p + \beta x'$, the amplitude and phase of the subharmonic or superharmonic oscillation must satisfy:

$$2 \frac{n}{m} \omega \frac{dR}{d\tau} = \sum_m C^p_{m,n}(R, S) \sin m\phi - \frac{n}{m} \omega\beta R, \tag{41}$$

$$2 \frac{n}{m} \omega R \frac{d\phi}{d\tau} = \sum_m D^p_{m,n}(R, S) \cos m\phi + B(R, S) - \gamma R \tag{42}$$

where

$$D^p_{m,n}(R, S) = \frac{1}{m} \frac{\partial}{\partial R} [RC^p_{m,n}(R, S)]. \tag{43}$$

These expressions will be used for the special case $p = 5$ in the next section.

6. Subharmonics and superharmonics for the Duffing equation of degree five. It is instructive to consider in detail the case $p = 5$. The formula cited in the previous section shows there to be three possible subharmonics, $m/n = \frac{5}{1}, \frac{3}{1}, \frac{4}{2}$. There are no repeated pairs of m and n which give the same ratio, so the summations in (41) and (42) reduce to a single term for each subharmonic. Equilibrium synchronized solutions are given by $dR/d\tau = 0$ and $d\phi/d\tau = 0$ or

$$\begin{aligned} C^p_{m,n}(R, S) \sin m\phi - \frac{n}{m} \omega\beta R &= 0, \\ D^p_{m,n}(R, S) \cos m\phi + B(R, S) - \gamma R &= 0. \end{aligned} \tag{44}$$

Independent of the value of m, the functions $B(R, S)$, $C^p_{m,n}(R, S)$ and $D^p_{m,n}(R, S)$ are homogeneous polynomials in the R and S variables of degree p with exponent increments of 2. This suggests that the Eqs. (44) be divided by γR and new variables be introduced:

$$X_{n/m} = \left(\frac{R}{\gamma^{1/p-1}}\right)^2, \qquad Y_{n/m} = \left(\frac{R}{\gamma^{1/p-1}}\right)^2, \qquad \alpha = \frac{n}{m} \omega \frac{\beta}{\gamma} \quad . \tag{45}$$

so that (44) simplifies to

$$\frac{C^p_{m,n}}{R} \sin m\phi - \alpha = 0, \qquad \frac{D^p_{m,n}}{R} \cos m\phi + \frac{B}{R} - 1 = 0 \tag{46}$$

where the arguments of all functions are $X_{n/m}$ and $Y_{n/m}$.

In the absence of dissipation, $\beta = 0 = \alpha$, and the equilibrium solutions are given simply by

$$\sin m\phi = 0, \qquad \frac{D^{v}_{m,n}}{R} = \pm(1 - B/R). \tag{47}$$

The solution of these equations is displayed in Fig. 1. It should be noted that $X_{n/m} = 0$

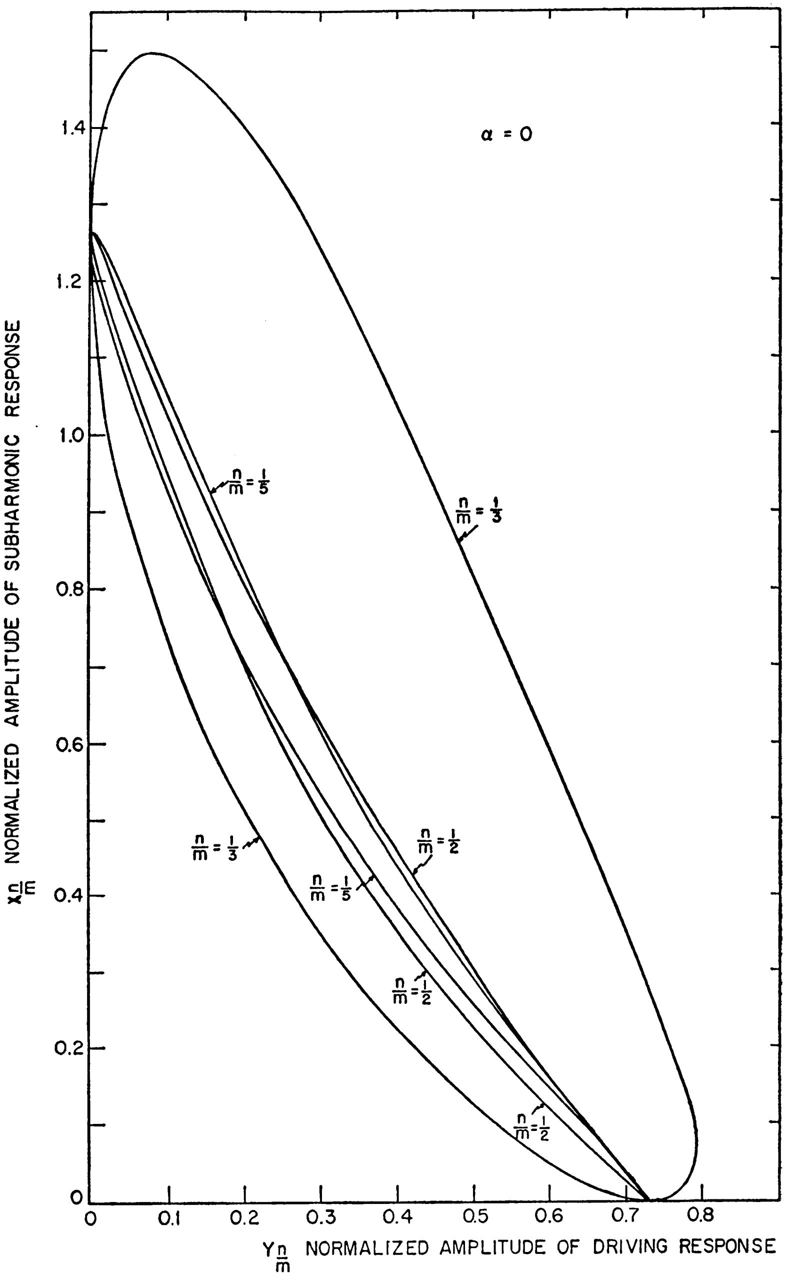

FIG. 1 AMPLITUDE RESPONSE CURVES FOR SUBHARMONIC OSCILLATIONS OF FIFTH ORDER NONLINEARITY.

is always a solution to (46) so that at any value of $Y_{n/m}$ there are either one or three possible values of $X_{n/m}$. If dissipation is included we find a family of curves for values of $\alpha > 0$, as shown in Fig. 2 for $n/m = \frac{1}{5}$. There is a maximum value of α for which the curve shrinks to a point, and for larger dissipation the subharmonic cannot be sustained.

With the equilibrium points of (41) and (42) in hand we can proceed to analyze other features of the phase portrait of these equations. The stability of equilibrium can be studied by linearizing (41) and (42) about the equilibrium point in question (S and ω are held fixed while R and ϕ assume small increments). This analysis is given in [8] and will be omitted here. The result is that all of the points above the locus of vertical tangents in Fig. 2 are stable foci or nodes, while those below are saddles. For selected values of $Y_{n/m}$ and α of Fig. 2 we have the phase portraits of Fig. 3. In the case $\alpha = 0$ (no dissipation) the points A and D of Fig. 2 are the center and saddle, respectively of Fig. 3a. Note that in this limiting case no real synchronization is possible and the subharmonic oscillation is either:

(a) above equilibrium amplitude with continually increasing phase,
(b) below equilibrium amplitude with continually decreasing phase,
(c) near equilibrium amplitude with oscillating phase and amplitude.

For $\alpha > 0$ the points B and C of Fig. 2 are the focus and saddle of Fig. 3b. Any initial conditions lying in the shaded regions will lead to synchronization at the $\frac{1}{5}$ subharmonic. This subharmonic is "hard excited" since no small initial disturbances will lead to synchronization. All initial conditions outside the shaded regions lead ultimately to zero amplitude subharmonic—or pure harmonic oscillations at the driving frequency. Phase portraits like these were also obtained by Hayashi [2].

By considering the initial phase $\phi(0)$ to be a random variable which might take on any value between 0 and 2π, it is possible to determine the probability of capture associated with each level of the initial amplitude of the subharmonic. The probability of capture is then the ratio between the phases which will lead to synchronization to the total possible phases 2π. Clearly, the probability of capture for the phase portrait in Fig. 3b increases with the amplitude for initial values less than that of the equilibrium state. It reaches a maximum when the initial value is in the neighborhood of the equilibrium state and decreases for higher amplitudes. For initial conditions which are far from the equilibrium solutions, the assumption of slow variations of the amplitude and phase no longer hold and the probability of capture cannot be determined with any accuracy. The greater the magnitude of α, the narrower is the separation between B and C and, hence, the smaller is the probability of capture. In other words, the range of initial conditions which will lead to synchronization is decreased, and also a system running synchronized at B is more susceptible to disturbances.

A more conventional display of synchronization properties is afforded if specific values are selected for the driving amplitude, F, and the strength of the nonlinearity, ϵ. Then the closed curves of Figure 1 may be transformed to the wedge-shaped bands of Figure 4. The $X_{n/m} = 0$ point of Fig. 1 transforms into the distinct $R = 0$ points of Fig. 4, and regions of small $Y_{n/m}$ become regions of high ω. Since the analysis of this paper has assumed γ to be of order unity the detuning, $(n^2/m^2)\omega^2 - \omega_0^2$, should be of order ϵ and the curves of Fig. 4 are valid only near the appropriate frequency. The dotted portions of the curves are large detuning behavior which has been predicted by

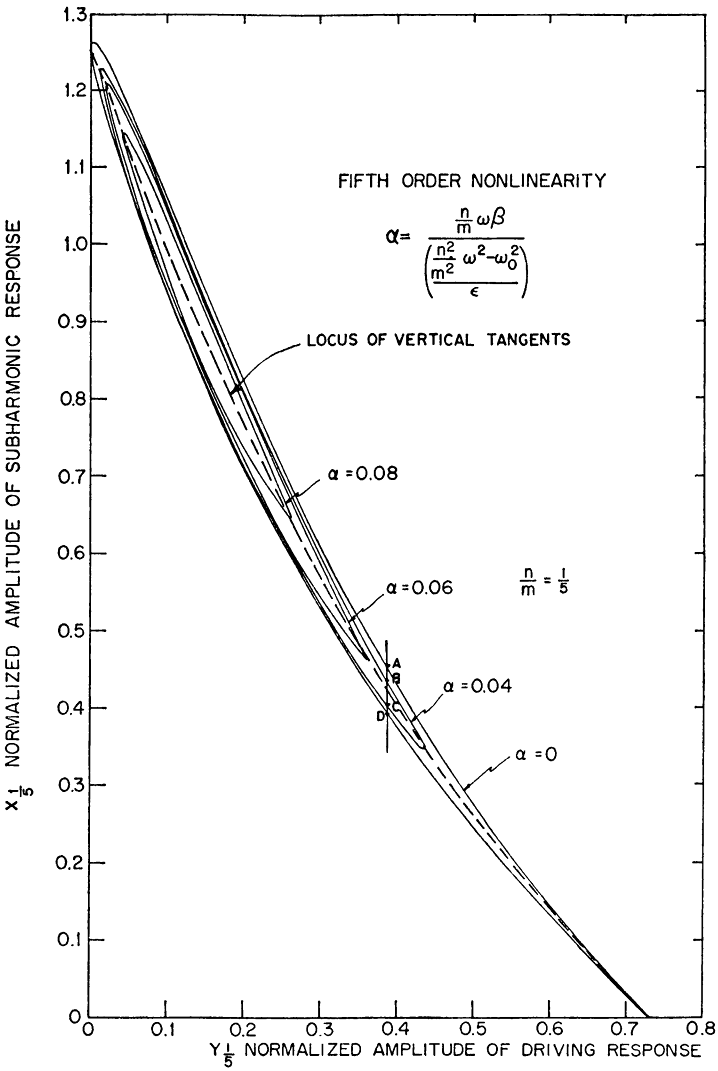

FIG. 2 AMPLITUDE RESPONSE CURVES FOR FIFTH ORDER SUBHARMONIC.

a small detuning theory and are therefore questionable. (An analysis of large detuning is given in [8].)

Figure 2 has shown that points for $\alpha > 0$ are interior to the $\alpha = 0$ curve. Similarly, curves for $\beta > 0$ are interior to the $\beta = 0$ curves of Fig. 4, although they have been omitted for clarity. It should be noted that the physically interesting case $\beta =$ constant is *not* a mapping of the curve $\alpha =$ constant of Fig. 2. The locus of vertical tangents which separated the stable from unstable singularities in Fig. 2 maps into Fig. 4 as a locus of vertical tangents for the curves $\beta =$ constant, and points above this locus are stable synchronized subharmonics.

Each order of subharmonic exists only in a single frequency band, and the width of each band is smaller as $n + m$ is larger. (Note that the width of $n + m = 2 + 4 = 6$

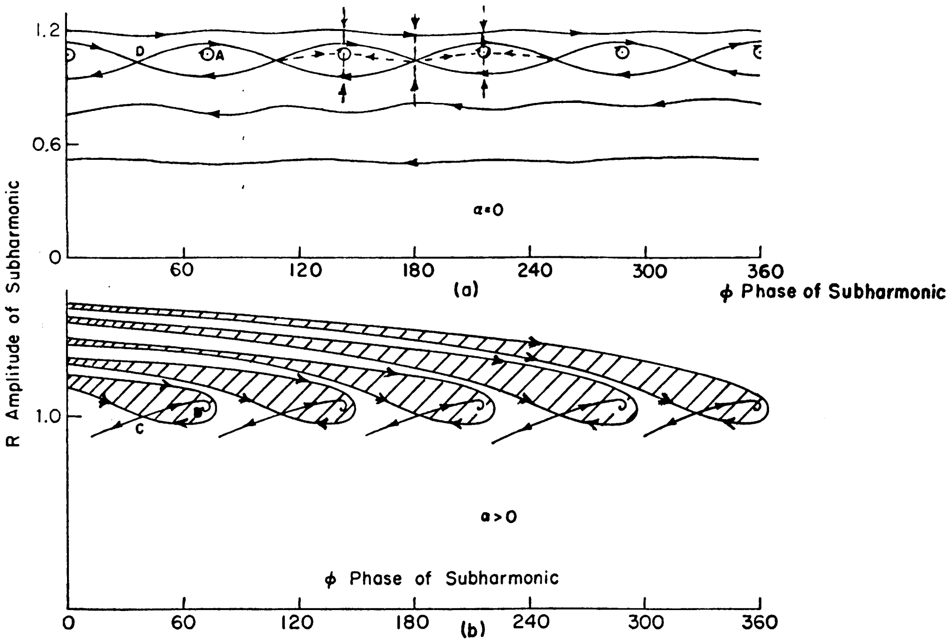

FIG. 3 TYPICAL INTEGRAL CURVES FOR FIFTH ORDER SUBHARMONIC

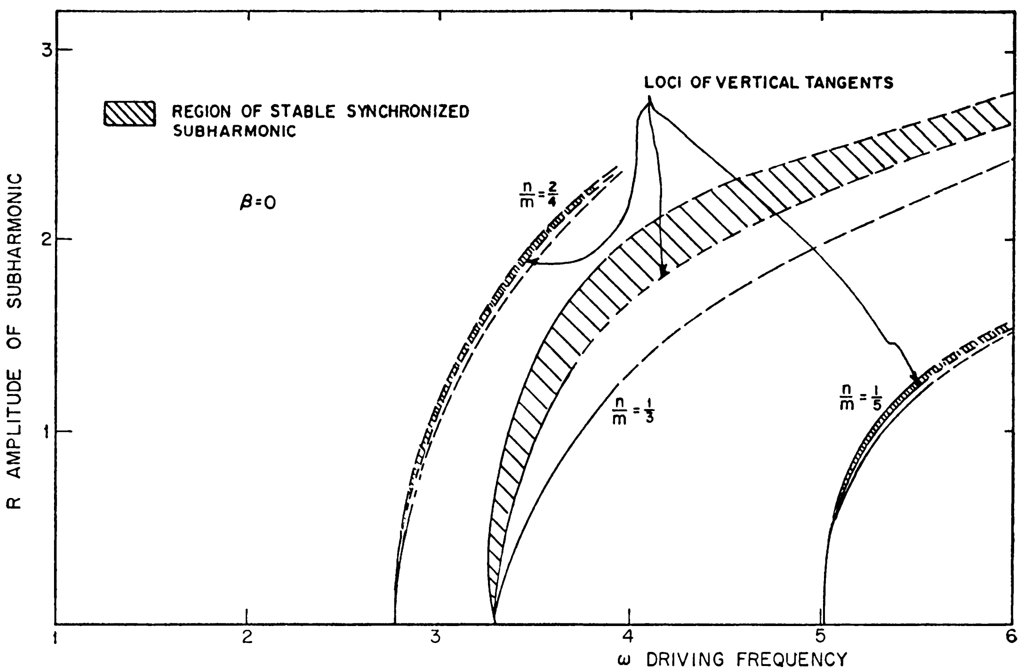

FIG. 4 SUBHARMONIC RESPONSE CURVES FOR FIFTH ORDER NONLINEARITY ($F=10.0, \epsilon=0.1$)

is very nearly equal to that of $n + m = 1 + 5 = 6$.) Due to the greater separation between stable and unstable equilibrium points of the $n/m = \frac{1}{3}$ subharmonic, it has, for a given dissipation the greatest probability of capture. This explains the relative ease of sustaining it experimentally.

So far the discussion has concerned subharmonics. For superharmonics much the same remarks apply. There are three superharmonic frequencies, and the variable transformation (45) is again employed. For the conservative case ($\beta = 0$) the equilibrium characteristics of Fig. 5 are analogous to Fig. 1. Interior to each pair of curves for a given n/m there are curves corresponding to values of $\alpha > 0$. These have not been shown. They are somewhat the same as the curves of Fig. 2 except that in this case they are open toward the $Y_{n/m}$ axis and there is no limit to the size of $\alpha > 0$ for which they exist. Again the locus of vertical tangents to these curves of constant α separates the stable singularities from the unstable.

For a specific choice of F and ϵ, Fig. 5 may be mapped into Fig. 6. For the superharmonic $n/m = 3$ the separation into stable and unstable synchronized solutions has been indicated. It is interesting to observe the jump phenomenon which these oscillations exhibit, similar to that well-known in harmonic synchronization. The corresponding hysteresis cycle for $n/m = 3$ is sketched in Fig. 6. If the driving frequency is swept slowly and continuously upward, R will follow the upper branch of the curve (a small $\beta > 0$ is assumed). When the point of vertical tangency is reached the oscillation drops abruptly. On decreasing the driving frequency there is a corresponding abrupt increase in the amplitude of R, but at a much lower frequency.

7. Generalized Duffing equation, even nonlinearity. The Duffing equation with $p =$ even merits special comment. The sub- and superharmonics which are able to be synchronized are shown in [8] to be

$$\frac{m}{n} = \frac{p + 1 - k - 2v}{k}, \qquad p = \text{even},$$

$$\text{where} \quad k = 1, 2, 3, \cdots p; \qquad v = 0, 1, 2, 3, \cdots \begin{cases} (p - k - 1)/2 & \text{for } k \text{ odd}, \\ (p - k)/2 & \text{for } k \text{ even}. \end{cases}$$

Equations (41) and (42) still apply, and $C^p_{m,n}$ and $D^p_{m,n}$ are again given by (39), (40) and (43). An important difference is that now $B = 0$. From this it follows (see [8]) that all synchronized solutions other than the trivial $R = 0$ are unstable (saddle type singularities). To the degree of approximation of this analysis no subharmonic or superharmonic oscillations can be synchronized with an even p. However, if the driving force were to contain an additional constant term of order unity, then the driven response would contain a corresponding bias displacement. All of the foregoing analysis can be made applicable by a simple shift of the datum of x. This shift of datum changes an even nonlinearity into a nonlinearity containing a mixture of even and odd terms. Now the asynchronous term, B, in the Eqs. (41) and (42) is restored and, curiously, synchronization is once again possible.

8. Example: Generalized Van der Pol equation. According to the results of Sec. 3, the analysis of the Van der Pol system follows from that of the Duffing system since the two are duals to each other. Therefore, if we consider the system:

$$x'' + \epsilon x'(px^{p-1} - \chi) + \omega_0^2 x = F \cos \omega t, \qquad F, \chi > 0, \qquad p = \text{odd} \tag{48}$$

then the amplitude and phase of the sub- or superharmonic oscillation $R \cos [(n/m)\omega t + \phi]$ satisfy:

$$2 \frac{n}{m} \omega \frac{dR}{d\tau} = -\frac{n}{m} \omega \Big[\sum_m D^p_{m,n}(R, S) \cos m\phi + B(R, S)\Big] + \frac{n}{m} \omega \chi R,$$
$$2 \frac{n}{m} \omega R \frac{d\phi}{d\tau} = \frac{n}{m} \omega \Big[\sum_m C^p_{m,n}(R, S) \sin m\phi\Big] - \mu R \tag{49}$$

where $\mu = [(n^2/m^2)\omega^2 - \omega_0^2)]/\epsilon$, and $B(R, S)$, $C^p_{m,n}(R, S)$ and $D^p_{m,n}(R, S)$ are defined by (38, 39, 40, 43).

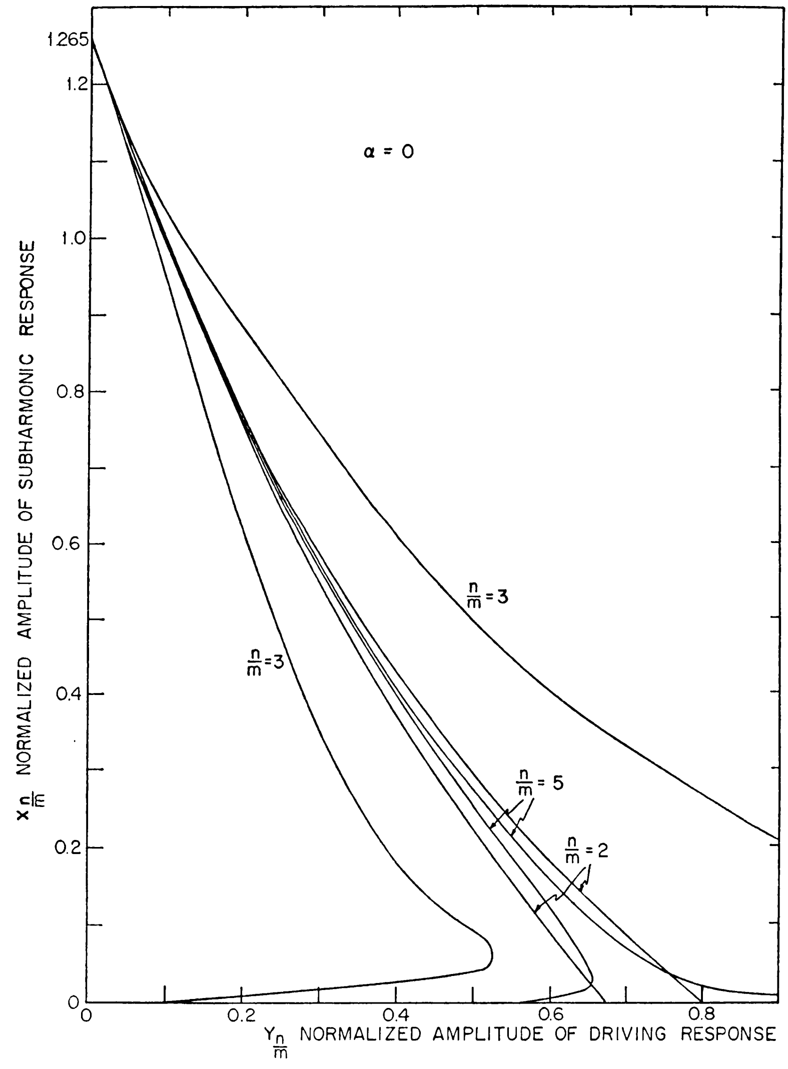

FIG. 5 AMPLITUDE RESPONSE CURVES FOR SUPERHARMONICS OF FIFTH ORDER NONLINEARITY.

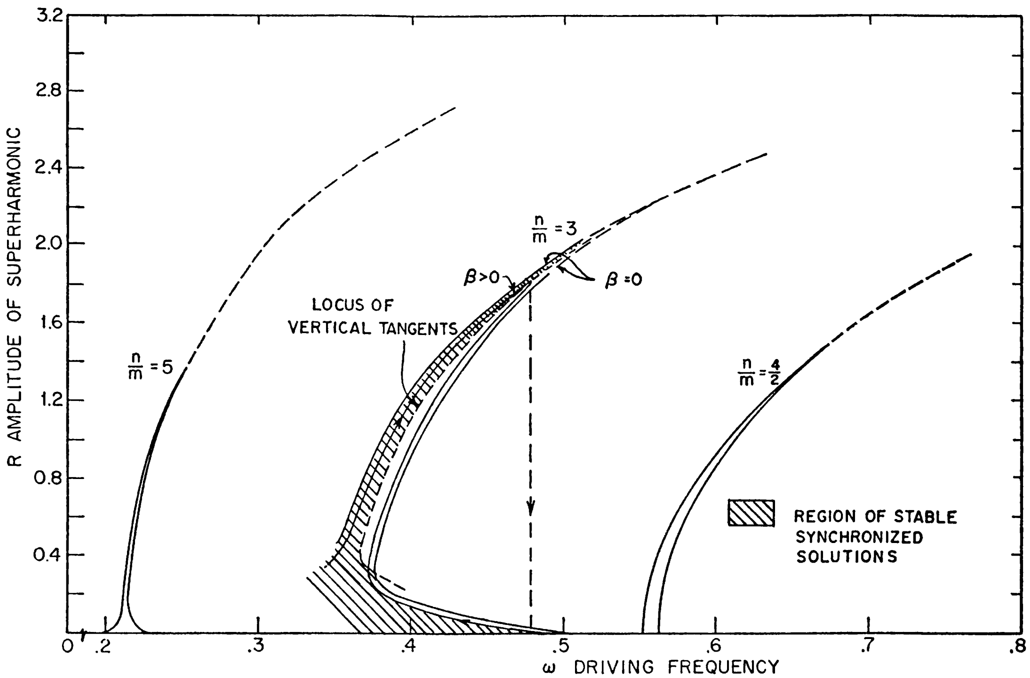

FIG. 6 SUPERHARMONIC RESPONSE CURVES FOR FIFTH ORDER NONLINEARITY ($F = 1.0$, $\epsilon = 0.1$).

Through the equivalence relationships (29) and (30), Figs. 1, 2 and 5 represent the subharmonic and superharmonic synchronized solutions in the self-excited (Van der Pol) system. The coordinates of these figures now have the following representations:

$$X_{n/m} = \frac{R^2}{(\chi)^{2/p-1}}\,; \qquad Y_{n/m} = \frac{S^2}{(\chi)^{2/p-1}}\,; \qquad \alpha = \frac{\mu}{(n/m)\omega\chi}. \tag{50}$$

Because of the relationship between singularity types in dual systems it follows that the locus of vertical tangents in Fig. 2 once again separates saddles (below) from the other types (above). It is interesting to note that in the Duffing type system it was necessary that the detuning parameter, γ, be different from zero for synchronized solutions (as $\gamma \to 0$, $Y_{n/m} \to \infty$). Here in the Van der Pol system the requirement is that χ be different from zero, i.e., that the self excitation at very small amplitude be above some threshold level.

The $R - \phi$ integral curves of the Van der Pol system are orthogonal to those of the dual Duffing system when plotted in log $R - \phi$ coordinates. For a limited range near $R = 1$ the orthogonality holds approximately in the $R - \phi$ plane. The Van der Pol trajectories are sketched in Fig. 3 for a small segment of the portrait.

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