CLASSICAL ANALYTIC REPRESENTATIONS*

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1. Summary. Mittag-Leffler-like fractional decompositions are constructed for the gamma, Jacobian elliptic and for the quotient of Bessel functions. These results illustrate a technique herein developed which permits a complete ML decomposition of a large class of meromorphic functions without resorting to any other previously derived information or knowledge of the particular function. Typically, the point of departure of this note stands in contrast to a statement in Knopp [1, p. 44], during his development of the ML decomposition of $\pi \cot \pi z$, which reads as follows: "The still undetermined entire function, $G(z)$, cannot be ascertained solely from the nature and position of the poles."

The new technique specifies conditions under which the "undetermined entire function" can be ascertained solely from the nature and position of the poles.

2. Theory. Suppose an arbitrary meromorphic function,

$$F(z) = \sum_{n=0}^{\infty} R_n(z) + \sum_{n=0}^{\infty} B_n Z^n \qquad (R_n(z) \text{ rational functions}),$$

has the following properties.

$$(1) \quad F\left(\frac{1}{z}\right) = \sum_{n=0}^{\infty} \frac{A_n}{z - (z_n)^{-1}} + \sum_{n=0}^{\infty} \frac{B_n}{z^n},$$

$$(2) \quad (a) \quad \sum_{n=0}^{\infty} A_n \quad \text{is convergent,} \qquad (b) \quad \sum_{n=0}^{\infty} \left|\frac{A_n}{z_n}\right| < \infty,$$

$z_k \neq 0$, $k = 0, 1, 2, \cdots$, are simple poles of $R_n(z)$; then

$$B_0 = \frac{1}{2\pi i} \int_c \frac{F(1/z)\, dz}{z},$$

$$B_n = \frac{1}{2\pi i} \int_c z^{n-1} F(1/z)\, dz - \sum_{k=1}^{\infty} \frac{A_k}{(z_k)^{n-1}},$$

$n = 1, 2, \cdots$; c is any Jordan curve containing all the poles and the nonisolated essential singularity of $F(1/z)$ in its interior.

$$B_n = \frac{1}{2\pi i} \int_c z^{n-1} F(1/z)\, dz - \sum_{k=1}^{\infty} \frac{A_k}{(z_k)^{n-1}}.$$

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Proof.

$$\int_c z^n \cdot \sum_{k=1}^{\infty} \frac{A_k}{z - (z_k)^{-1}}\, dz = \int_c [(z - (z_k)^{-1}) + (z_k)^{-1}]^n \cdot \left(\sum_{k=1}^{\infty} \frac{A_k}{z - (z_k)^{-1}} \right) dz \tag{1}$$

$$= \sum_{k=1}^{\infty} \int_c \frac{A_k z_k^{-n}\, dz}{z - (z_k)^{-1}} + \sum_{k=1}^{\infty} \sum_{j=1}^{n} \int_c A_k \binom{n}{j} (z - (z_k)^{-1})^{j-1} \cdot z_k^{j-n}\, dz \tag{2}$$

$$= \sum_{k=1}^{\infty} \int_c \frac{A_k z_k^{-n}\, dz}{z - (z_k)^{-1}} = 2\pi i \sum_{k=1}^{\infty} \frac{A_k}{z_k^n}\,; \qquad n = 0, 1, 2, \cdots . \tag{3}$$

This follows, first, from seeing that for $n \geqq 1$, hypothesis 2b implies the absolute convergence of $\sum_{k=1}^{\infty} A_k z_k^{-n}/(z - (z_k)^{-1})$ and of

$$\sum_{k=1}^{\infty} \sum_{j=1}^{n} \binom{n}{j} (z - (z_k)^{-1})^{j-1} \cdot A_k z_k^{j-n}\,;$$

clearly there must exist k_0 sufficiently large so that for each $n \geqq 1$,

$$\sum_{k=k_0}^{\infty} |A_k z_k^{-n}| = \sum_{k=k_0}^{\infty} |A_k| \cdot |z_k^{-n}| \leqq \sum_{k=k_0}^{\infty} |A_k| \cdot |z_k^{-1}| = \sum_{k=k_0}^{\infty} \left|\frac{A_k}{z_k}\right| < \infty . \tag{4}$$

For $n = 0$, hypotheses 2a and 2b implies convergence of $\sum_{k=1}^{\infty} A_k/(z - (z_k)^{-1})$; secondly, uniform convergence on c permits the term-by-term integration. Inasmuch as each integrand of the double sum is an integral function, the double sum vanishes. Finally, for $n = -1$ we have

$$\int_c \frac{1}{z} \sum_{k=1}^{\infty} \frac{A_k}{z - (z_k)^{-1}} = \sum_{k=1}^{\infty} \int_c \frac{A_k\, dz}{z(z - (z_k)^{-1})} = 0. \tag{5}$$

Turning now to the other part of $F(1/z)$, we have

$$\int_c z^n \sum_{k=0}^{\infty} \frac{B_k}{z^k}\, dz = \sum_{k=0}^{\infty} \int_c \frac{B_k\, dz}{z^{k-n}} = 2\pi i\, B_{n+1}\,, \tag{6}$$

$n = -1, 0, 1, 2, \cdots$; combining (3), (5), and (6), we have,

$$\int_c z^n F\left(\frac{1}{z}\right) dz = \begin{cases} 2\pi i \left(\sum_{k=1}^{\infty} A_k z_k^{-n} + B_{n+1} \right); & n = 0, 1, 2, \cdots . \\ 2\pi i B_0\,; & n = -1 \end{cases} \tag{7}$$

From (7), the representation follows.

3. Applications. We first develop the ML decomposition of $\Gamma(z)$ [2]; consider $\Gamma(1 + z)$, simple poles at $z = -k$, $k = 1, 2, \cdots$; $\Gamma(1 + 1/z)$ has simple poles at $(z_k)^{-1} = -1/k$, corresponding residues $(-1)^k/(k \cdot k!)$. With $A_k = (-1)^k/(k \cdot k!)$, $(z_k)^{-1} = -1/k$, hypotheses 2a, 2b, are seen to hold. Consequently,

$$\Gamma\left(1 + \frac{1}{z}\right) = \sum_{k=1}^{\infty} \frac{(-1)^k}{kk!\left(z + \frac{1}{k}\right)} + \sum_{n=0}^{\infty} \left[\frac{1}{2\pi i} \int_c z^{n-1} \Gamma\left(1 + \frac{1}{z}\right) dz - \sum_{k=1}^{\infty} \frac{(-1)^{k+n-1}}{k^n \cdot k!} \right] \cdot \frac{1}{z^n} .$$

For c any contour meeting the conditions of the theory,

$$\frac{1}{2\pi i} \int_c z^{n-1} \Gamma\left(1 + \frac{1}{z}\right) dz = \frac{\Gamma^{(n)}(1)}{n!} = (-1)^n \cdot n! \sum_{k=1}^{n} \frac{T^{k,n}(1)}{k!}\,; \qquad n = 0, 1, 2, \cdots ;$$

$$T^{k,n}(1) = \sum \frac{S_{r_1} S_{r_2} \cdots S_{r_k}}{r_1 \cdot r_2 \cdots r_k} .$$

The sum is taken over all positive integers, $r_1, r_2, \cdots r_k$, which satisfy the unique condition $r_1 + r_2 + \cdots + r_k = n$, with care being taken that an appointed combination of k numbers which satisfy the condition $r_1 + r_2 + \cdots + r_k = n$ should be calculated as many times as is possible to form different permutations of these k numbers without repeating them [3].

$$S_1 = \lim_{n\to\infty}\left[1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right],$$

$$S_n = \left[1 + \frac{1}{2^n} + \frac{1}{3^n} + \cdots + \frac{1}{k^n} + \cdots\right] \qquad \text{for } n \geqq 2,$$

$$\Gamma\left(1 + \frac{1}{z}\right) = \sum_{k=1}^{\infty} \frac{(-1)^k}{kk!\,(z + 1/k)} + 1 + \sum_{n=1}^{\infty}\left[(-1)^n \sum_{k=1} \frac{T^{k,n}(1)}{k!} + \sum_{k=1}^{\infty} \frac{(-1)^{n+k}}{k^n k!}\right]\cdot\frac{1}{z^n}.$$

Replacing $1/z$ by z and recalling that $\Gamma(1 + z) = z\Gamma(z)$,

$$\Gamma(z) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\,(z + k)} + \sum_{n=1}^{\infty}\left[(-1)^n \sum_{k=1}^{\infty} \frac{T^{k,n}(1)}{k!} + \sum_{k=1}^{\infty} \frac{(-1)^{n+k}}{k^n\cdot k!}\right]\cdot z^n.$$

This representation of $\Gamma(z)$ specifies explicitly the poles, residues and additive integral function as a power series. It may be compared with Prym's integral representation

$$\Gamma(z) = \int_0^1 t^{z-1}e^{-t}\,dt + \int_1^{\infty} t^{z-1}e^{-t}\,dt.$$

We may now represent $\int_1^{\infty} t^{z-1}e^{-t}\,dt$ as a Taylor series,

$$\sum_{r=0}^{\infty} a_r(z - 1)^r, \qquad |z - 1| < 1\cdot$$

From the series representation we see that

$$\Gamma^{(r)}(1) = \sum_{k=0}^{\infty} \frac{(-1)^{k+r}r!}{(k + 1)\,!\,(k + 1)^r} + r! \sum_{n=r}^{\infty}\left[(-1)^n \sum_{k=1}^{\infty} \frac{T^{k,n}(1)}{k!} + \sum_{k=1} \frac{(-1)^{n+k}}{k^n k!}\right], \qquad r \geqq 1;$$

hence

$$a_0 = 1; \qquad a_r = \frac{\Gamma^{(r)}(1)}{r!} \quad r \geqq 1.$$

We now derive the ML representation for the Jacobian elliptic function, sn $(z) =$ sn (z, k), for which, "except in a certain special case" [4], the additive entire function is unknown. This function is doubly periodic with periods $4K$ and $2iK'$; its poles are simple and located at the points $z_{m,n} = 2mK + (2n + 1)iK'$. The residue at $z = z_{m,n}$ is

$$(-1)^m k^{-1}.$$

Here m and n assume, independently, all integral values including $m = n = 0$. K and K' are elliptic complete normal integrals expressible by theta null series or as hypergeometric series in terms of the modulus k, $k \neq \pm 1$. The only property used in the following is that the imaginary part of iK'/K is positive.

The ML theorem applied to sn (z) now gives

$$\text{sn}\,(z) = \sum_{m,n} \frac{(-1)^m}{k}\left\{\frac{1}{z - z_{m,n}} + \frac{1}{z_{m,n}} + \frac{z}{(z_{m,n})^2}\right\} + \sum_{r=0}^{\infty} c_r z^r\cdot \tag{1}$$

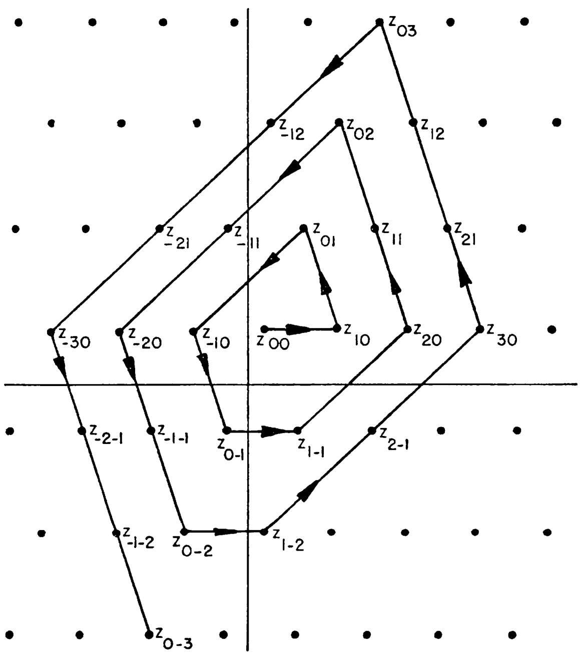
The c_r can now be determined by the following variation of the new technique. We adopt the following ordering of the poles to meet hypotheses 2a, 2b (see graph below).

(a) The first pole $z_{0,0}$ lies inside all parallelograms;

(b) all other poles lie on parallelograms such that $|m| + |n| = p$ for the pth parallelogram, $p = 1, 2, \cdots$;

(c) the ordering on any parallelogram is counterclockwise;

(d) when going from one pole to the next on any parallelogram m and n change by ± 1 producing alternating signs for the residues within any parallelogram; this is also true when going from the last pole on one to the first pole of the next. There are an even number, $4p$, poles on the pth parallelogram.



THE ORDERING

Consider now the function sn $(1/z)$. It has poles at $z = (z_{m,n})^{-1}$ with corresponding residues $(-1)^{m+1}k^{-1}(z_{m,n})^{-2}$. With m and n defined as before, and restricting the order of summation to that of the parallelogram spiral, we have to prove the convergence of the sum of residues. Simple geometric considerations show the existence of a fixed positive number d such that

$$|z_{m,n}| > dp, \qquad |m| + |n| = p, \quad p > p_0 .$$

Next, two consecutive terms of the series give

$$\pm k^{-1}[(z_{m,n})^{-2} - (z_{h,j})^{-2}]$$

where $|m - h| = 1$, $|n - j| = 1$. The bracket equals

$$(z_{h,j} - z_{m,n})(z_{h,j} + z_{m,n})(z_{m,n})^{-2}(z_{h,j})^{-2}.$$

Here

$$z_{h,j} - z_{m,n} = 2(h - m)K + 2(j - n)iK',$$
$$z_{h,j} + z_{m,n} = 2(h + m)k + 2(j + n + 1)iK'.$$

The absolute value of the differences does not exceed

$$2(|K| + |K'|) \equiv C$$

and the sum does not exceed $(2p + 1)C$. Hence

$$|(z_{h,j})^{-2} - (z_{m,n})^{-2}| \leqq C^2 d^{-4}(2p + 1)p^{-4}, \qquad p > p_0 .$$

On the pth parallelogram there are $2p$ such differences so their total contribution to the sum of the series does not exceed a constant times p^{-2}. This proves the convergence of the series

$$k^{-1} \sum_{m,n} (-1)^m (z_{m,n})^{-2} \equiv R$$

when summed in the manner indicated. This shows that condition 2a is satisfied.

Condition 2b requires the absolute convergence of the series

$$\sum_{m,n} (z_{m,n})^{-3}.$$

That this holds is well known [5].

Applying the technique, we have

$$\operatorname{sn}\left(\frac{1}{z}\right) = \sum_{m,n} \frac{(-1)^{m+1}}{k} \frac{1}{(z_{m,n})^2[z - (z_{m,n})^{-1}]} + \sum_{r=0}^{\infty} \frac{B_r}{z^r} \tag{2}$$

where the series is summed in the "spiral" order. Replacing $1/z$ by z we have

$$\operatorname{sn}(z) = \sum_{m,n} \frac{(-1)^m}{k} \left[\frac{1}{z - z_{m,n}} + \frac{1}{z_{m,n}}\right] + \sum_{r=0}^{\infty} B_r z^r \tag{3}$$

where

$$B_0 = \frac{1}{2\pi i} \int_c \frac{1}{t} \operatorname{sn}\left(\frac{1}{t}\right) dt,$$

$$B_r = \frac{1}{2\pi i} \int_c t^{r-1} \operatorname{sn}\left(\frac{1}{t}\right) dt - \sum_{m,n} (-1)^m k^{-1} (z_{m,n})^{-r-1}.$$

Here c is any circle of radius greater than the reciprocal of the distance of the set $z_{m,n}$ from the origin. In the formula for B_r the series is absolutely convergent if $r > 1$ while for $r = 1$ it converges to the sum R if summed in the spiral manner.

Set

$$\operatorname{sn}(z) = \sum_{j=0}^{\infty} (-1)^j a_j z^{2j+1}.$$

Here $a_0 = 1$ and the a_j with $j > 0$ are polynomials in k^2 with positive coefficients. This gives

$$B_0 = 0, \qquad B_1 = 1 - R, \qquad B_{2s} = 0, \quad s = 1, 2, \cdots$$

where we have used the fact that if $r = 2s$ then the integral is zero as well as the series, the latter since $z_{-m,-n-1} = z_{m,n}$ for all m, n. Actually $B_r = 0$ for all $r > 1$. To prove this we return to the expansion (3) above.

Set

$$S_p(z) = \sum_{|m|+|n|\leq p}\sum \frac{(-1)^m}{k}\left[\frac{1}{z - z_{m,n}} + \frac{1}{z_{m,n}}\right]$$

and

$$\lim_{p\to\infty} S_p(z) = S(z).$$

We shall prove that $S(z)$ is doubly periodic with the same periods, $4K$ and $2iK'$, as sn (z). To this end, take P odd and form

$$S_p(z + 4K) - S_p(z) = \sum_{|m|+|n|\leq p}\sum \frac{(-1)^m}{k}\left[\frac{1}{z - z_{m-2,n}} - \frac{1}{z - z_{m,n}}\right].$$

In this finite sum the majority of terms cancel and we are left with $(p + 1)$ terms from each of the parallelograms numbered $p - 1$, p, $p + 1$, $p + 2$. These terms have alternating residues and form $2(p + 1)$ pairs of the form

$$\pm\frac{1}{k}\left[\frac{1}{z - z_{m,n}} - \frac{1}{z - z_{h,j}}\right]$$

where, as above, $|m - h| = 1$, $|n - j| = 1$. For $z \in D$, a compact set of positive distance from the poles, the absolute value of this difference is $O(p^{-2})$. There are $2(p + 1)$ such differences. Hence

$$S_p(z + 4K) - S_p(z) = O(p^{-1}) \to 0$$

so that

$$S(z + 4K) = S(z).$$

In the same manner one shows that

$$S(z + 2iK') = S(z).$$

Thus $S(z)$ and sn (z) have the same poles, the same residues and the same periods. It follows that sn $(z) - S(z)$ is a doubly periodic function without poles and hence a constant. This constant must be zero since sn $(0) = S(0) = 0$.

We have thus proved that

$$\text{sn}\ (z) = S(z)$$

and this shows that

$$B_r = 0, \quad \text{all} \quad r.$$

In particular,

$$R = 1.$$

Since the ML expansion for sn (z) differs from $S(z)$ by $Rz = z$ we get, finally,

$$\text{sn}\ (z) = \sum_{m,n} \frac{(-1)^m}{k}\left[\frac{1}{z - z_{m,n}} + \frac{1}{z_{m,n}} + \frac{z}{(z_{m,n})^2}\right] - z. \tag{3}$$

Incidentally, the formulas for B_r show that

$$(-1)^i a_i = \frac{1}{k} \sum_{m,n} (-1)^m (z_{m,n})^{-2i}.$$

The technique will apply equally well to all the explicit variations of the elliptic family such as quotients of theta functions, etc.

Finally, we use the technique to decompose the quotient of Bessel functions into a ML representation, as well-known result derived first by Watson [6], using classical residue theory. The technique will then be brought into play in conjunction with Watson's result to evaluate in closed form sums of negative powers of the zeros of $J_n(z)$ [7].

Consider

$$F(z) = \frac{J_{n+1}(z)}{J_n(z)} \quad \text{for } n \geqq 0,$$

$$J_n(z) = \sum_{r=0} \frac{(-1)^r \left(\frac{z}{2}\right)^{n+2r}}{r!\,(n+r)\,!}, \qquad n = 0, 1, 2, \cdots;$$

$z_k = \pm j_{n,k} \cong (r + \frac{3}{4} + n/2)\pi$, $r = 0 \pm 1 \pm 2, \cdots$; for large n [6]. $f(z) = (1/z)F(1/z) = J_{n+1}(1/z)/zJ_n(1/z)$ has simple poles at $z = \pm 1/j_{n,k}$, corresponding residues $1/\pm j_{n,k}$;

Hypotheses 2a, 2b in section 2 are now met if we order the poles by sign and magnitude.

$$\left(\frac{1}{j_{n,1}} \quad \frac{1}{j_{n,1}}\right) + \left(\frac{1}{j_{n,2}} - \frac{1}{j_{n,2}}\right) + \cdots \text{ converges} \tag{2a}$$

$$2 \sum_{k=1}^{\infty} \frac{1}{(j_{n,k})^2} < \infty. \tag{2b}$$

We have immediately by the technique that

$$f(z) = \sum_{k=1}^{\infty} \frac{1}{j_{n,k}} \left[\frac{1}{z - (j_{n,k})^{-1}} - \frac{1}{z + (j_{n,k})^{-1}}\right] + \sum_{r=0}^{\infty} \frac{B_r}{z^r},$$

$$B_0 = \frac{1}{2\pi i} \int_c \frac{f(z)\,dz}{z}, \tag{1}$$

$$B_r = \frac{1}{2\pi i} \int_c z^{r-1} f(z)\,dz - \left[\left(\frac{1}{j_{n,1}}\right)^r + \left(\frac{-1}{j_{n,1}}\right)^r + \left(\frac{1}{j_{n,2}}\right)^r + \left(\frac{-1}{j_{n,2}}\right)^r + \cdots\right] \qquad r \geqq 1$$

$$f\left(\frac{1}{z}\right) = \frac{zJ_{n+1}(z)}{J_n(z)} = 2z^2 \sum_{k=1}^{\infty} \frac{1}{(j_{n,k})^2 - z^2} + \sum_{r=0}^{\infty} B_r z^r$$

$$\frac{J_{n+1}(z)}{J_n(z)} = 2z \sum_{k=1}^{\infty} \frac{1}{(j_{n,k})^2 - z^2} + \sum_{r=0}^{\infty} B_r z^{r-1}. \tag{3}$$

Watson finds that

$$\frac{J_{n+1}(z)}{J_n(z)} = 2z \sum_{k=1}^{\infty} \frac{1}{(j_{n,k})^2 - z^2}. \tag{4}$$

In view of (3) and (4) we see that $B_r \equiv 0$. The formal series expansion for $f(z)$ turns out to be

$$f(z) = \frac{1}{(2n+2)z^2}\left(1 + \frac{1}{(2n+2)(2n+4)z^2} + \frac{1}{(n+1)(2n+2)(2n+4)(2n+6)z^4} + \cdots\right)$$

It follows that, $B_0 = B_1 = B_{2k+1} = 0$,

$$B_2 = \frac{1}{2n+2} - 2\sum_{k=1}^{\infty}\left(\frac{1}{j_{n,k}}\right)^2 = 0 \Rightarrow \sum_{k=1}^{\infty}\left(\frac{1}{j_{n,k}}\right)^2 = \frac{1}{2(2n+2)},$$

$$B_4 = \frac{1}{(2n+2)^2(2n+4)} - 2\sum_{k=1}^{\infty}\left(\frac{1}{j_{n,k}}\right)^4 = 0 \Rightarrow \sum_{k=1}^{\infty}\left(\frac{1}{j_{n,k}}\right)^4 = \frac{1}{2(2n+2)^2(2n+4)}.$$

Clearly, $\sum_{k=1}^{\infty} (1/j_{n,k})^{2r} = \frac{1}{2}$ (coefficient of z^{2r}) in the series expansion of $(1/z)F(1/z)$; these results were derived a number of years ago using M. Riesz's typical means.

4. Extensions and observations. 1. Multiple poled meromorphic functions and functions with mixed poles and *finitely many* isolated essential singularities, assuming proper convergence restrictions, may be represented in this way.

2. Functions with a finite set of nonisolated essential singularities may be represented in this way.

3. Functions with divergent "principal parts" become increasingly difficult for the technique to manage as the number of terms of the "convergence factors" increase.

4. Many times, a convenient ordering of the B_k, z_k (as shown for sn (z) and $J_n(z)$) will induce convergence.

5. The technique suggests a new area of divergent series theory: the theory of divergent series of complex rational functions.

6. The technique is powerful in evaluating closed form sums of constants and functions. It would have summed many series of the like that occupied Watson, Hardy, Ramanujan, etc. early in the century.

7. The new representations have proven to be very well adapted to computer applications.

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