DECAY OF THE KINETIC ENERGY OF MICROPOLAR INCOMPRESSIBLE FLUIDS*

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The equations of multipolar continuum mechanics have been the subject matter of study in recent years by Eringen [1] Green [2] and others (*vide* the references cited in the above two papers). The linear constitutive equation for these fluids leads to an interesting theory in which micro-rotational effects and couple stress are prevalent. Such a linear theory of micropolar incompressible fluids has also been considered by Eringen [3] and Bleustein and Green [4]. In this theory the fluid motion is characterized by two vector fields $\mathbf{V}$ and $\boldsymbol{\nu}$ representing respectively the velocity of flow and the micro-rotation. The field equations of this theory are [3]

$$\partial \rho/\partial t + \operatorname{div} \rho \mathbf{V} = 0, \tag{1}$$

$$\rho d\mathbf{V}/dt = \rho \mathbf{f} - \operatorname{grad} p + K \operatorname{curl} \boldsymbol{\nu} - (\mu + k) \operatorname{curl} \operatorname{curl} \mathbf{V} + (\lambda + 2\mu + k) \operatorname{grad} (\operatorname{div} \mathbf{V}), \tag{2}$$

$$\rho j d\boldsymbol{\nu}/dt = \rho \iota + k (\operatorname{curl} V - 2\boldsymbol{\nu}) - \gamma \operatorname{curl} \operatorname{curl} \boldsymbol{\nu} + (\alpha + \beta + \gamma) \operatorname{grad} (\operatorname{div} \boldsymbol{\nu}). \tag{3}$$

The constants λ, μ, k are viscosity coefficients while α, β, γ and the gyration parameter j are other constants of the fluid. These conform to the inequalities

$$3\lambda + 2\mu + k \geq 0, \quad \mu \geq 0, \quad k \geq 0, \quad \gamma \geq 0, \quad |\beta| \leq \gamma, \quad 3\alpha + \beta + \gamma \geq 0 \tag{4}$$

and it follows that

$$\lambda + 2\mu + k \geq 0, \qquad \alpha + \beta + \gamma \geq 0. \tag{5}$$

It has been noticed by Leray, Kampé de Fériet et al. that the kinetic energy of the Navier–Stokes viscous liquid in a domain with rigid walls decays. Kampé de Fériet [5] proved that for the Navier–Stokes fluids, the decay of the kinetic energy is faster than the exponential. In the present note we obtain the corresponding result for micropolar fluids.

Let R be a domain in space bounded by the regular curve Γ and let the vectors $\mathbf{V}$, $\boldsymbol{\nu}$ possess continuous second order derivatives in R and vanish on Γ. The kinetic energy of the fluid is

$$T = (\rho/2) \int \mathbf{V}^2 \, d\tau + (\rho j/2) \int \boldsymbol{\nu}^2 \, d\tau. \tag{6}$$

The integrals in Eq. (6) and everywhere else in this note are over the volume of R, the only exception being in Eq. (17). We have the inequalities

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$$T_1 = (\rho/2) \int \mathbf{V}^2 \, d\tau \leq (\rho\Lambda/8\pi) \int (\text{curl } \mathbf{V})^2 \, d\tau \tag{7}$$

$$T_2 = (\rho j/2) \int \mathbf{v}^2 \, d\tau < (\rho j\Lambda/8\pi) \int \{(\text{curl } \mathbf{v})^2 + (\text{div } \mathbf{v})^2\} \, d\tau \tag{8}$$

in which the constant Λ depends only on the geometry of the domain and is defined by

$$\Lambda^2 = \iint (PQ)^{-2} \, d\tau_P \, d\tau_Q \, . \tag{9}$$

If the domain can be included in a ball of diameter d, the constant coefficients multiplying the integrals on the right side in Eqs. (7), (8) can be replaced by $\rho d^2/(3 + 13^{1/2})\pi^2$ and $\rho j d^2/6\pi^2$ respectively, as pointed out by Serrin [6].

From Eqs. (6), (7), (8) we have

$$T = T_1 + T_2 \leq (\rho\Lambda/8\pi) \int \{(\text{curl } \mathbf{V})^2 + j(\text{curl } \mathbf{v})^2 + j(\text{div } \mathbf{v})^2\} \, d\tau. \tag{10}$$

Assuming that the body force is derivable from a potential field, we have

$$\rho(\partial \mathbf{V}/dt - \mathbf{V} \times \text{curl } \mathbf{V}) = \text{grad } F + k \text{ curl } \mathbf{v} - (\mu + k) \text{ curl curl } \mathbf{V}$$

and

$$\begin{aligned} \frac{dT_1}{dt} &= \int \rho \mathbf{V} \cdot \frac{\partial \mathbf{V}}{\partial t} \, d\tau \\ &= k \int \mathbf{V} \cdot \text{curl } \mathbf{v} \, d\tau - (\mu + k) \int \mathbf{V} \cdot \text{curl curl } \mathbf{V} \, d\tau \qquad (11) \\ &= k \int \mathbf{V} \cdot \text{curl } \mathbf{v} \, d\tau - (\mu + k) \int (\text{curl } \mathbf{V})^2 \, d\tau. \end{aligned}$$

Omitting the body couple in Eq. (3), we get

$$\rho j \frac{\partial \mathbf{v}}{\partial t} + \rho j(\mathbf{V} \cdot \text{grad})\mathbf{v} = -2k\mathbf{v} + k \text{ curl } \mathbf{V} - \gamma \text{ curl curl } \mathbf{v} + (\alpha + \beta + \gamma) \text{ grad } (\text{div } \mathbf{v}).$$

From this we can see that

$$\begin{aligned} \frac{dT_2}{dt} = \rho j \int \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t} \, d\tau = -2k \int \mathbf{v}^2 \, d\tau \\ + k \int \mathbf{v} \cdot \text{curl } \mathbf{V} \, d\tau - \gamma \int (\text{curl } \mathbf{v})^2 \, d\tau - (\alpha + \beta + \gamma) \int (\text{div } \mathbf{v})^2 \, d\tau. \qquad (12) \end{aligned}$$

From Eqs. (11) and (12) we get

$$\begin{aligned} \frac{d}{dt}(T_1 + T_2) = \frac{dT}{dt} = -\mu \int (\text{curl } \mathbf{V})^2 \, d\tau - k \int \mathbf{v}^2 \, d\tau - \gamma \int (\text{curl } \mathbf{v})^2 \, d\tau \\ - (\alpha + \beta + \gamma) \int (\text{div } \mathbf{v})^2 \, d\tau - k \int (\mathbf{v} - \text{curl } \mathbf{V})^2 \, d\tau \qquad (13) \end{aligned}$$

from which the decreasing nature of T is evident. Using the bounds given in Eqs. (7) and (8) for T_1 and T_2 in Eq. (13), we get

$$\frac{dT}{dt} \leq -(8\pi\mu/\rho\Lambda)T_1 - [(2k/\rho j) + (8\pi a/\rho j\Lambda)]T_2$$

in which a is a positive number equal to min $(\alpha + \beta + \gamma, \gamma)$. If b denotes the positive number equal to minimum $[\mu, \{(a/j) + (k\Lambda/4\pi j)\}]$, we see that

$$dT/dt \leq -(8\pi b/p\Lambda)T \tag{14}$$

and now it follows that

$$T(t) \leq T(t_0) \exp[-(8\pi b/p\Lambda)(t - t_0)]. \tag{15}$$

The decay of the kinetic energy is faster than the exponential rate. It would be of interest to examine if the velocity and micro-rotation also decay in this manner.

The spectral function of the kinetic energy also decreases faster than the exponential. Let $r = (x, y, z)$ and $\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3)$ denote the position vector in the space of the fluid and in the space Ω of the real variables $\omega_1, \omega_2, \omega_3$. If $\mathbf{V} = (u, v, w)$ and $\boldsymbol{\nu} = (A, B, C)$ are the velocity and micro-rotation components, we define

$$U(\omega_1, \omega_2, \omega_3, t), \quad V(\omega_1, \omega_2, \omega_3, t), \quad W(\omega_1, \omega_2, \omega_3, t),$$
$$X(\omega_1, \omega_2, \omega_3, t), \quad Y(\omega_1, \omega_2, \omega_3, t), \quad Z(\omega_1, \omega_2, \omega_3, t)$$

to be their Fourier transforms over the domain R. We have thus

$$U(\omega_1, \omega_2, \omega_3, t) = (8\pi^3)^{-1} \int u(x, y, z, t) \exp[i(\boldsymbol{\omega}\cdot r)]\, d\tau \tag{16}$$

and the inverse relation is

$$u(x, y, z, t) = \int U(\omega_1, \omega_2, \omega_3, t) \exp[-i(\boldsymbol{\omega}\cdot r)]\, d\boldsymbol{\omega} \tag{17}$$

The integral in Eq. (17) is over the entire space spanned by $\omega_1, \omega_2, \omega_3$. The spectral function $\gamma(\omega_1, \omega_2, \omega_3, t)$ of the kinetic energy is seen to be

$$\gamma(\omega_1, \omega_2, \omega_3, t) = 4\pi^3\rho\{|U|^2 + |\mathbf{V}|^2 + |W|^2 + j(|X|^2 + |Y|^2 + |Z|^2)\}. \tag{18}$$

From Schwarz's inequality in Eq. (16) we get

$$(8\pi^3 |U|)^2 \leq (\text{Vol. } R) \int u^2\, d\tau. \tag{19}$$

From inequalities of this type for the Fourier transforms we see that

$$\gamma(\omega_1, \omega_2, \omega_3, t) \leq (8\pi^3)^{-1}(\text{Vol. } R)T \tag{20}$$

and now it is clear that

$$\gamma(\omega_1, \omega_2, \omega_3, t) \leq (8\pi^3)^{-1}(\text{Vol. } R)T(t_0)\cdot\exp[-(8\pi b/\rho\Lambda)(t - t_0)] \tag{21}$$

on using Eq. (15).

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