THE PAUCITY OF UNIVERSAL MOTIONS IN THERMOELASTICITY*

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1. Introduction. The work presented here is motivated by the work of Ericksen [1], [2] in the classical theory of elasticity for compressible and incompressible bodies. Ericksen was concerned with finding universal static deformations in homogeneous isotropic hyperelastic bodies in the absence of body forces. He showed [2] that the only deformations which can be maintained in all homogeneous compressible isotropic hyperelastic bodies under the action of surface forces alone are necessarily homogeneous. For incompressible bodies the general solution is not complete. In addition to homogeneous deformations and the four families found by Ericksen, a further example has been exhibited by Klingbeil and Shield [3] and Singh and Pipkin [4].

For compressible bodies the corresponding dynamic problem is trivial—the motions must be homogeneous and accelerationless. The incompressible case appears to be nontrivial. Some discussion and universal solutions of this problem may be found in the book of Truesdell and Noll [5].

The problem set here is to find what sort of motions and temperature fields are possible in every homogeneous isotropic thermoelastic body in the absence of body forces and external heat supply. We consider isotropic thermoelastic bodies and suppose that at some initial time the configuration of each body is homogeneous and undistorted. An elementary analysis permits us to show that universal motions must be rigid motions with the temperature constant in space and time. To be more specific, suppose that initially each body is at rest. Then there are no universal motions of compressible bodies and the only universal motions of incompressible bodies are pure translations.

It is worthy of note that the universal motions obtained here apply to all thermoelastic bodies which are initially homogeneous.

2. Preliminaries. We use the terminology and, with some minor differences, the notation of Truesdell and Noll [5]. In this paper we always use the initial configuration as the reference configuration. Let $\mathbf{x}$ be the spatial position, at time t, of the material particle whose position in the configuration at time $t = 0$ is $\mathbf{X}$. If θ is the absolute temperature, assumed to be positive, then a motion of the body and its temperature are given by

$$\mathbf{x} = \mathbf{x}(\mathbf{X}, t), \qquad \theta = \hat{\theta}(\mathbf{X}, t).$$

The deformation gradient is denoted by $\mathbf{F}$ so that the left Cauchy–Green tensor is $\mathbf{B} = \mathbf{F}\mathbf{F}^T$. The principal invariants of B are

$$I = \mathrm{tr}\,\mathbf{B}, \qquad 2II = [(\mathrm{tr}\,\mathbf{B})^2 - \mathrm{tr}\,\mathbf{B}^2], \qquad III = \det \mathbf{B}.$$

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We denote the Cauchy stress tensor by $\mathbf{T}$ and use a superposed dot to indicate material time differentiation. In the absence of body forces the equations of motion are

$$\operatorname{div} \mathbf{T} = \rho \ddot{\mathbf{x}}, \tag{2.1}$$

where ρ is the density and div denotes the spatial divergence operator. Let $\mathbf{h}$ be the heat flux vector and η the specific entropy; then for compressible or incompressible thermoelastic bodies with no external heat supply, the energy equation becomes

$$\rho\theta\dot{\eta} - \operatorname{div} \mathbf{h} = 0. \tag{2.2}$$

In this paper we only consider isotropic thermoelastic bodies and assume that at $t = 0$ the configuration of each body is homogeneous and undistorted. For our purpose it is easier to consider compressible bodies and incompressible bodies separately. Of course, we must assume that the constitutive functions are sufficiently smooth.

3. Compressible bodies. Let ψ denote the specific free energy; then for compressible bodies

$$\psi = \hat{\psi}(I, II, III, \theta). \tag{3.1}$$

Also

$$\mathbf{T} = \beta_0 \mathbf{1} + \beta_1 \mathbf{B} + \beta_2 \mathbf{B}^2,$$

where

$$\beta_0 = 2\rho III\, \partial\hat{\psi}/\partial III, \qquad \beta_1 = 2\rho(\partial\hat{\psi}/\partial I + I\, \partial\hat{\psi}/\partial II), \qquad \beta_2 = -2\rho\, \partial\hat{\psi}/\partial II.$$

The specific entropy is determined through

$$\eta = -\partial\hat{\psi}/\partial\theta. \tag{3.2}$$

If $\mathbf{g}$ denotes the spatial temperature gradient grad θ, then

$$\mathbf{h} = (\alpha_0 \mathbf{1} + \alpha_1 \mathbf{B} + \alpha_2 \mathbf{B}^2)\mathbf{g} \equiv \mathbf{A}\mathbf{g}, \tag{3.3}$$

where α_0, α_1, α_2 are functions of θ and the invariants

$$I,\ II,\ III,\ \mathbf{g}\cdot\mathbf{g},\ \mathbf{g}\cdot\mathbf{B}\mathbf{g},\ \mathbf{g}\cdot\mathbf{B}^2\mathbf{g}.$$

The Clausius–Duhem inequality demands that $\mathbf{g}\cdot\mathbf{A}\mathbf{g} \geq 0$ for all $\mathbf{g}$.

We wish to determine the most general class of motions and temperature distributions in every body of the class designated above. With such a wide class of functions $\hat{\psi}$, α_0, α_1, α_2 at our disposal it is not surprising that the possible solutions are few. First, we substitute (3.1) and (3.2) into (2.2):

$$\rho\theta\left(\frac{\partial^2\hat{\psi}}{\partial\theta^2}\,\dot{\theta} + \frac{\partial^2\hat{\psi}}{\partial I\,\partial\theta}\,\dot{I} + \frac{\partial^2\hat{\psi}}{\partial II\,\partial\theta}\,\dot{II} + \frac{\partial^2\hat{\psi}}{\partial III\,\partial\theta}\,\dot{III}\right) + \operatorname{div}\mathbf{h} = 0.$$

Since the constitutive functions are arbitrary, we must have

$$\dot{\theta} = 0, \qquad \dot{I} = \dot{II} = \dot{III} = 0, \tag{3.4}$$

and

$$\operatorname{div}\mathbf{h} = 0. \tag{3.5}$$

Now make the choice $\alpha_0 = 1$, $\alpha_1 = \alpha_2 = 0$, in (3.3). Then (3.5) gives

$$\operatorname{div}(\operatorname{grad}\theta) = 0. \tag{3.6}$$

Next put $\alpha_0 = \theta$, $\alpha_1 = \alpha_2 = 0$, in (3.3). Then from (3.5)

$$\operatorname{div}(\theta \operatorname{grad}\theta) = 0. \tag{3.7}$$

Combining (3.6) and (3.7) we conclude[3] that grad $\theta = 0$. Now by (3.4), the temperature θ at any particle is constant in time. Thus it follows that the temperature is constant in space and time.

Returning to $(3.4)_2$ it is seen that I, II and III are, at any particle, constant in time. However, the deformation gradient has been computed taking the configuration at $t = 0$ as reference, so

$$I = 3, \qquad II = 3, \qquad III = 1, \tag{3.8}$$

for all time and every particle. With the help of the Cayley–Hamilton theorem,

$$\mathbf{B}^3 - I\mathbf{B}^2 + II\mathbf{B} - III\mathbf{1} = \mathbf{0},$$

we see that (3.8) implies that

$$\mathbf{B} = \mathbf{1} \tag{3.9}$$

for every particle for all time. Hence the configurations of the body for $t > 0$ are obtained from the configuration at $t = 0$ by rigid rotations and translations:

$$\mathbf{x} = \mathbf{Q}(t)\mathbf{X} + \mathbf{c}(t), \tag{3.10}$$

where $\mathbf{Q}(t)$ is orthogonal. Substituting these results into the equations of motion (2.1) and remembering that $\mathbf{x} = \mathbf{X}$ when $t = 0$, we find that (3.10) must reduce to

$$\mathbf{x} = (\mathbf{1} + t\mathbf{Q}_0)\mathbf{X} + t\mathbf{c}_0\,, \tag{3.11}$$

where $\mathbf{c}_0$ and $\mathbf{Q}_0$ are constant and $(\mathbf{1} + t\mathbf{Q}_0)$ is an orthogonal tensor. Since $(\mathbf{1} + t\mathbf{Q}_0)$ is orthogonal for all t, we deduce that

$$\mathbf{Q}_0 = \mathbf{0}.$$

Thus the class of univeral motions is, at most, given by the constant velocity motions

$$\mathbf{x} = \mathbf{X} + t\mathbf{c}_0\,. \tag{3.12}$$

However, any pure translation (3.12) is obviously a universal motion, so that the class of universal motions is the class of constant velocity pure translations. Evidently for bodies initially at rest no universal motions are possible.

4. Incompressible bodies. For incompressible bodies there is a material constraint

$$III = 1 \tag{4.1}$$

and instead of (3.1) we have

$$\psi = \bar{\psi}(I, II, \theta). \tag{4.2}$$

The stress is determined through

[3]In their paper on thermostatics, Petroski and Carlson [6] also concluded from (3.5) that grad $\theta = 0$.

$$\mathbf{T} = -p\mathbf{1} + \beta_1\mathbf{B} + \beta_2\mathbf{B}^2,$$

where

$$\beta_1 = 2\rho(\partial\bar{\Psi}/\partial I + I\,\partial\bar{\Psi}/\partial II), \qquad \beta_2 = -2\rho\,\partial\bar{\Psi}/\partial II \tag{4.3}$$

and p is an indeterminate pressure. We have

$$\eta = -\partial\bar{\Psi}/\partial\theta, \tag{4.4}$$

and the heat flux is given by (3.3) but this time α_0, α_1, α_2 are functions of θ and

$$I,\ II,\ \mathbf{g}\cdot\mathbf{g},\ \mathbf{g}\cdot\mathbf{B}\mathbf{g},\ \mathbf{g}\cdot\mathbf{B}^2\mathbf{g}.$$

We proceed as in the previous section and substitute (4.2) and (4.4) into (2.2) to get

$$\rho\theta\left(\frac{\partial^2\bar{\Psi}}{\partial\theta^2}\,\dot{\theta} + \frac{\partial^2\bar{\Psi}}{\partial I\,\partial\theta}\,\dot{I} + \frac{\partial^2\bar{\Psi}}{\partial II\,\partial\theta}\,\dot{II}\right) + \operatorname{div}\mathbf{h} = 0.$$

As before, we deduce that

$$I = 3,\ II = 3, \tag{4.5}$$

and that θ is constant in time and space. From (4.1), (4.5) and the Cayley–Hamilton theorem we again conclude that $\mathbf{B} = \mathbf{1}$ for every particle for all time. Thus the only possible universal motions must be of the form

$$\mathbf{x} = \mathbf{Q}(t)\mathbf{X} + \mathbf{c}(t), \tag{4.6}$$

where $\mathbf{Q}(t)$ is orthogonal. However, not every motion of the type (4.6) is possible since the equations of motion (2.1) still need to be satisfied for motions of the form (4.6), that is

$$-\operatorname{grad} p = \rho\ddot{\mathbf{x}}. \tag{4.7}$$

Now we have already assumed that the body is homogeneous at $t = 0$ so ρ is simply a constant. Hence, from (4.7) we need

$$\operatorname{grad}\ddot{\mathbf{x}} = (\operatorname{grad}\ddot{\mathbf{x}})^T. \tag{4.8}$$

If we now use (4.6) and remember that $\mathbf{Q}(t)$ is orthogonal, we see that (4.8) implies that

$$\dot{\mathbf{Q}}(t)\mathbf{Q}(t)^T = \mathbf{W} = -\mathbf{W}^T, \tag{4.9}$$

where $\mathbf{W}$ is a constant antisymmetric tensor. At time $t = 0$, $\mathbf{x} = \mathbf{X}$, so we have

$$\mathbf{Q}(0) = \mathbf{1}, \qquad \mathbf{c}(0) = \mathbf{0}. \tag{4.10}$$

The solution of the differential equation (4.9) subject to the initial condition $(4.10)_1$ is $\mathbf{Q}(t) = e^{t\mathbf{W}}$. Thus the only possible universal motions must be of the reduced form

$$\mathbf{x} = e^{t\mathbf{W}}\mathbf{X} + \mathbf{c}(t), \tag{4.11}$$

with $\mathbf{c}(0) = \mathbf{0}$. One can readily show that (4.11) is a possible motion of every incompressible isotropic body and that the pressure is given by

$$p = \rho\mathbf{x}\cdot(-\tfrac{1}{2}\mathbf{W}^2\mathbf{x} + \mathbf{W}^2\mathbf{c}(t) - \ddot{\mathbf{c}}(t)) + p_0,$$

where p_0 is an arbitrary function of time. Hence, (4.11) gives the complete set of universal motions of incompressible isotropic thermoelastic bodies.

Suppose now that the initial velocity is zero; then $\mathbf{W} = \mathbf{0}$, $\dot{\mathbf{c}}(0) = \mathbf{0}$, and the only universal motions are pure translations.

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